SOME INTERESTING LOGARITHMIC INEQUALITIES AND A PROBLEM OF PÓLYA AND SZEGÖ

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Abstract. The results of Problem 168 of Chapter I from the well-known "Problems and Theorems in Analysis" of Pólya and Szegö are extended by showing that the sequence $\{(1+1/n)^{n+p}\}$ increases if and only if $p \le c_1 = -1 + \ln(3/2)/(\ln(2) - \ln(3/2)) \le .409\ldots$, and by determining the length of the initial decreasing sequence between c_1 and $1/2$. Some related inequalities are also discussed.

1. Introduction. Problem 168 of Chapter I from the book of Pólya and Szegö [3] states the following:

Problem 168. *The sequence*

$$a_n = (1 + \frac{1}{n})^{n+p}, \quad n = 1, 2, \ldots \tag{1}$$

is monotone decreasing if, and only if $p \ge 1/2$.

The solution presented in [3] is due to I. Schur, and it is also proved in [3] that $\{a_n\}$ increases if n is larger than a certain subscript n_0 when $p < 1/2$, and the entire sequence increases when $p \le 0$. However, Schur's solution does not indicate how the subscript n_0 should be chosen. Also it does not describe the behavior of the initial subsequence up to n_0. It will be shown in the third section of this paper that (1) is increasing if and only if $p \le c_1 := -1 + \ln(3/2)/(\ln(2) - \ln(3/2)) = .409\ldots$, and the length of the initial subsequence up to n_0, which is decreasing, will also be determined for $c_1 < p < 1/2$. This description will be made with the aid of the sequence

$$c_i := -i + \frac{\ln(\frac{i+2}{i+1})}{\ln(\frac{i+1}{i}) - \ln(\frac{i+2}{i+1})} \quad i = 1, 2, \ldots \tag{2}$$

The elements of the sequence

$$b_i := \frac{1}{\ln(1 + \frac{1}{i})} - i, \quad i = 1, 2, \ldots \tag{3}$$

will be used, among other things, to derive some properties of sequence (2). Suitable functions will be introduced to generate sequences (2) and (3) and to obtain their required properties.

Let

$$F(x,y) := \frac{y\ln(1+\frac{1}{y}) - x\ln(1+\frac{1}{x})}{\ln(1+\frac{1}{x}) - \ln(1+\frac{1}{y})} \quad \text{for } x,y>0 \text{ with } x\neq y. \tag{4}$$

Notice that for fixed x and y with $0<x<y$,

$$\{p : (1+\frac{1}{x})^{x+p} \geq (1+\frac{1}{y})^{y+p}\} = \{p : p \geq F(x,y)\}. \tag{5}$$

Therefore, it is natural to investigate the properties of the symmetric function F for positive values of x and y. To simplify this process, consider first the function

$$f(x) := \lim_{y\to\infty} F(x,y) = \frac{1}{\ln(1+\frac{1}{x})} - x,\ x>0. \tag{6}$$

It follows at once from (2), (3), (4) and (6) that

$$c_i = F(i,i+1) \quad \text{and} \quad b_i = f(i),\ i=1,2,\ldots \tag{7}$$

The following well-known inequalities [2, p. 273] are essential for the proofs of the required properties of f.

$$\frac{2}{2x+1} < \ln(1+\frac{1}{x}) < \frac{1}{\sqrt{x^2+x}}, \quad x>0. \tag{8}$$

In the next section some interesting properties of f and the sequence $\{b_i\}$ will be presented. In the last section some related inequalities will be discussed. Also the results of Problem 94 of Chapter 3 from [1] will be extended.

2. Logarithmic inequalities. Some properties of f will be derived first.

Theorem 1. *Let f be defined by (6). Then f is strictly increasing, strictly concave and positive on* $(0,\infty)$ *with* $\lim_{x\to 0+} f(x) = 0$. *In addition,*

$$\begin{array}{ll} (i) & \displaystyle\lim_{x\to\infty} f(x) = \frac{1}{2}, \\ (ii) & \displaystyle\lim_{x\to\infty} xf'(x) = 0, \\ (iii) & \displaystyle\lim_{x\to\infty} x(f(x+1)-f(x)) = 0. \end{array}$$

Proof. It follows directly from the right-hand inequality of (8) that $f(x)>0$ for $x>0$. Since

$$f'(x) = \frac{1}{\ln^2(1+\frac{1}{x})x(x+1)} - 1, \tag{9}$$

the right-hand inequality of (8) also implies that $f'(x)>0$ for $x>0$. Thus f is strictly increasing and hence it is injective. By a simple computation we have that

$$f''(x) = -\frac{(2x+1)\ln^2(1+\frac{1}{x}) - 2\ln(1+\frac{1}{x})}{[x(x+1)\ln^2(1+\frac{1}{x})]^2}. \tag{10}$$

Now, (10) shows via the left-hand inequality of (8) that $f''(x)<0$ for $x>0$. Hence f is strictly concave on $(0,\infty)$. Clearly, $\lim_{x\to 0+} f(x) = 0$ and (i) can be verified by L'Hôpital's rule or by series expansion. Therefore, f has an inverse function with domain $(0,1/2)$. Using (9) together with the left-hand inequality of (8), we see that

$$0 < xf'(x) < \frac{(2x+1)^2}{4(x+1)} - x = \frac{1}{4(x+1)},$$

which proves (ii). Further, the Mean Value Theorem for f on the interval $[x,x+1]$ for $x>0$ yields

$$x\,(f(x+1)-f(x)) = xf'(\alpha(x)), \quad \text{for a certain } \alpha(x)\in(x,x+1).$$

Since f' is positive and decreasing, part (iii) now follows from (ii).

Corollary. *The following limit is a direct consequence of Theorem 1, part (iii):*

$$\lim_{i\to\infty} i(b_{i+1} - b_i) = 0. \tag{11}$$

The previous analysis also implies that

$$\{x : f(x) \le b_i\} = \{x : x \le f^{-1}(b_i) = i\}. \tag{12}$$

The assertion of the next result follows from Theorem 1 and from (12).

Theorem 2. *Sequence (3) is concave and monotonically increasing, with* $\lim_{i\to\infty} b_i = 1/2$*. Furthermore, for all* $i \in \mathbb{N}$

$$\ln(1 + \frac{1}{x}) \le \frac{1}{b_i + x} \quad \text{for } x \ge i, \tag{13}$$

while

$$\ln(1 + \frac{1}{x}) \ge \frac{1}{b_i + x} \quad \text{for } 0 < x \le i, \tag{14}$$

and b_i *is the largest positive number for which the inequality (13) holds on the indicated interval.*

Notice that $b_i \to 1/2$ can be found implicitly in [1, p. 274]. However this sequence was not introduced there.

3. The main result. First the properties of f are used to investigate F by expressing it in terms of f.

Theorem 3. *Let* F *be defined by (4). Then*

$$\begin{array}{lll} (i) & F(x,y) = \dfrac{yf(x) - xf(y)}{y - x + f(y) - f(x)} & \text{for all } x, y > 0 \text{ with } x \ne y, \\ (ii) & F(x,y) < \min\{f(x), f(y)\} & \text{for all } x, y > 0 \text{ with } x \ne y, \\ (iii) & F(x,y) < F(y,z) & \text{for all } x, y, z \text{ with } 0 < x < y < z. \end{array}$$

Proof. Part (i) follows immediately from the definition. Let $0 < x < y$. Because f is increasing, the inequality

$$\frac{yf(x) - xf(y)}{y - x + f(y) - f(x)} < f(x) \tag{15}$$

can be easily checked. The fact that F is symmetric together with (15) establishes (ii).

Let $0 < x < y < z$. Then $y = \lambda x + (1 - \lambda)z$ for a certain $\lambda \in (0, 1)$. As f is strictly concave, $f(y) > \lambda f(x) + (1 - \lambda)f(z)$ and this yields the inequality

$$\Sigma_{cycl} := (x - z)f(y) + (y - x)f(z) + (z - y)f(x) < 0. \tag{16}$$

From part (i) and (16) we deduce that

$$F(x,y) - F(y,z) = \frac{(y + f(y))\Sigma_{cycl}}{(y - x + f(y) - f(x))(z - y + f(z) - f(y))} < 0,$$

which completes the proof of part (iii).

Remark. *Using the Mean Value Theorem, it can be shown that the following limit*

$$L_a := \lim_{\substack{(x,y)\to(a,a) \\ x\neq y}} F(x,y)$$

exists for all $a \in [0,\infty]$. *In particular,* $L_0 = 0$, $L_\infty = 1/2$ *and*

$$L_a = \frac{f(a) - af'(a)}{1 + f'(a)} \quad \text{for } a \in (0,\infty).$$

Since we do not need it in the sequel, the verification is left to the reader.

From (7) and Theorem 3 part (i), it is easy to see that

$$c_i = F(i, i+1) = \frac{b_i + i(b_i - b_{i+1})}{1 + b_{i+1} - b_i}, \quad i = 1, 2, \ldots \tag{17}$$

It also follows from part (iii) of Theorem 3 that $\{c_i\}$ increases, and (17) shows by (11) and Theorem 2 that $c_i \to 1/2$. One can conclude by (ii) of Theorem 3 that

$$c_i = F(i, i+1) < \min\{f(i), f(i+1)\} = \min\{b_i, b_{i+1}\} = b_i.$$

Therefore the proof of the following theorem is complete.

Theorem 4. *Sequence (2) increases, with* $\lim_{i\to\infty} c_i = 1/2$. *Furthermore* $c_i < b_i$ *for all* $i \in \mathbb{N}$.

Substituting $x = i$ and $y = i + 1$ into (5) yields

$$\{p : a_i \geq a_{i+1}\} = \{p : (1 + \frac{1}{i})^{i+p} \geq (1 + \frac{1}{i+1})^{i+1+p}\} = \{p : p \geq F(i, i+1)\} = \{p : p \geq c_i\}.$$

The next result now follows from the previous discussion.

Theorem 5. *Sequence (1) is increasing if and only if* $p \leq c_1$, *and it is decreasing if and only if* $p \geq 1/2$. *When* $c_i \leq p \leq c_{i+1}$ *for* $i \in \mathbb{N}$ *then* $a_1 \geq a_2 \geq \ldots \geq a_{i+1}$, *while the sequence* $\{a_n\}$ *is increasing for* $n \geq i+1$.

4. Some further inequalities. In this section some related inequalities are derived.

Theorem 6. *Let* p *be a fixed real number. Then*

$$\begin{aligned}
(1 + \frac{1}{n})^{n+p} &\leq e \quad \text{for all } n \in \mathbb{N} \text{ when } p \leq b_1, \\
(1 + \frac{1}{n})^{n+p} &\geq e \quad \text{for } n = 1, \ldots, i \text{ when } b_i < p \leq b_{i+1}, \\
(1 + \frac{1}{n})^{n+p} &\leq e \quad \text{for } n \geq i+1 \text{ when } b_i < p \leq b_{i+1}, \\
(1 + \frac{1}{n})^{n+p} &\geq e \quad \text{for all } n \in \mathbb{N} \text{ when } p \geq \frac{1}{2}.
\end{aligned}$$

Proof. In the case when $p \leq c_1$ or $p \geq 1/2$ the assertion of this theorem follows immediately from Theorem 5. Therefore, without loss of generality, it can be assumed that $p \in (0, 1/2)$, which is the domain of the inverse function of f. Then

$$\{p : (1 + \frac{1}{i})^{i+p} \geq e\} = \{p : p \geq f(i)\} = \{p : p \geq b_i\} = \{p : f^{-1}(p) \geq i\},$$

and the conclusion follows.

Theorem 6 extends Problem 94 of Chapter 3 from the book of Gabriel Klambauer [1], which asks the following:

Problem 94. *Find the maximum value of α and the minimum value of β for which*

$$(1+\frac{1}{n})^{n+\alpha} \le e \le (1+\frac{1}{n})^{n+\beta}$$

for all positive integers n.

Finally, another related inequality is presented.

Theorem 7. *Let m and k be fixed positive integers and let r be an integer so that $m = 4k + r$ with $-2 \le r \le 1$. Let i and j be positive integers such that $1 \le i < j \le k$, then*

$$\sqrt{\frac{j}{i}} \ge (1+\frac{2}{m-1})^{j-i}.$$

Proof. Let $\ell = j - i$, hence $1 \le \ell \le k-1$. It suffices to show that

$$\sqrt{\frac{k}{k-l}} = \sqrt{1+\frac{l}{k-l}} \ge (1+\frac{2}{m-1})^{\ell}$$

Equivalently, it is sufficient to prove that

$$\ln(1+\frac{1}{\frac{k}{\ell}-1}) \ge 2\ell \ln(1+\frac{1}{\frac{m-1}{2}}).$$

Since $m \ge 2$, inequality (13) yields

$$\ln(1+\frac{1}{\frac{m-1}{2}}) \le \frac{1}{\frac{m-1}{2}+b_1}. \tag{18}$$

The left-hand inequality of (8) and (18) imply that it is enough to show the inequality

$$\frac{1}{\frac{k}{\ell}-\frac{1}{2}} \ge \frac{2\ell}{\frac{m-1}{2}+b_1}$$

or, equivalently, $m - 1 + 2b_1 \ge 4k - 2\ell$. This last inequality is true in all cases except when both $r = -2$ and $\ell = 1$. Therefore, it remains to be shown only that

$$\sqrt{\frac{k}{k-1}} \ge 1 - \frac{2}{4k-3} \quad \text{for } k \in \mathbb{N},\ k \ge 2.$$

Since this last inequality holds, the proof is complete.

References

1. G. Klambauer, *Problems and Propositions in Analysis*, Marcel Dekker, New York, 1979.
2. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1972.
3. G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, New York, 1972.