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THE DIOPHANTINE EQUATION

$(x^2+y)(x+y^2) = N(x-y)^3$

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ABSTRACT

In this note we prove that for any rational integer $N \neq 0$ the title equation has finitely many solutions in non-zero rational integers, by giving a rather small upper-bound for the size of the solutions. Moreover, we show that there are infinitely many values of N for which the equation has at least six solutions. We also give a table with complete sets of solutions for every N in the range $1 \leq N \leq 51$.

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1. INTRODUCTION

Let $(x,y) \in \mathbb{Z}^2$ be a solution of the equation

$$(x^2+y)(x+y^2) = N(x-y)^3, \quad N \in \mathbb{Z}\setminus\{0\} \qquad \ldots\ldots(1)$$

If $xy = 0$, then $|N| = 1$ and consequently $xy \neq 0$ for any N with $|N| \geq 2$. A solution (x,y) of (1) with $xy \neq 0$ will be called a *proper solution*. It is easy to see that equation (1) is always solvable: $x = y = -1$ gives a solution for every N. We shall refer to this solution as the *trivial solution* of (1).

In this paper we investigate the proper non-trivial solutions of equation (1). Our main result is laid down in the following theorem.

THEOREM 1 *If* (x,y) *is a proper solution of* (1), *then*

$$\max(|x|,|y|) < |N|^3,$$

provided $|N| \neq 1,2$ *or* 4. *In the exceptional cases we have*

$$\max(|x|,|y|) = 21, 54 \text{ and } 90$$

for $|N| = 1,2$ *and* 4 *respectively.*

The proof we shall give is elementary. We do not claim the upper-bound given in the theorem to be best possible.

There is no loss of generality to consider only positive values of N. Indeed, if $N < 0$, we need only interchange x and y to obtain an equivalent equation with a positive constant N.

At the end of this paper we have added a table of all proper non-trivial solutions of (1) for the values of N in the range $1 \leqq N \leqq 51$. We are very grateful to Mr. C.W.J.B. Slik for the assistance he rendered in compiling this table.

2. AUXILIARY RESULTS

In this section we prove some lemmas which are useful in the proof of THEOREM 1.

<u>LEMMA 1</u> *Let* (x,y) *be a non-trivial solution of* (1) *with* $N > 0$. *Then there are rational integers* u,v *and* $\ell \neq 0$ *such that*

$$2x = v-u+\ell+1 \ , \ 2y = v-u-\ell+1 \qquad \text{......(2)}$$

$$uv = N\ell \qquad \text{......(3)}$$

and

$$(u+v-\ell)^2 = 4(u-1)(v+1)+1 \qquad \text{......(4)}$$

Conversely, for every choice of u *and* v *satisfying* (3) *and* (4) *for some* $\ell \neq 0$, *the relations* (2) *define a non-trivial solution* (x,y) *of* (1) *with* $N > 0$.

Proof. Suppose that (x,y) is a non-trivial solution of (1) with $N > 0$. Put $k := x^2+y$, $\ell := x-y$ and $m := x+y-1$. Then $x+y^2 = k-m\ell$. Since (x,y) is non-trivial, $\ell \neq 0$. We have

$$2x = \ell+m+1 \ , \ 2y = -\ell+m+1 \qquad \text{......(5)}$$

and

$$4k = (\ell+m)^2+4m+3 \qquad \text{......(6)}$$

Moreover, equation (1) becomes in terms of k,ℓ and m

$$k^2-k\ell m-N\ell^3 = 0 \qquad \text{......(7)}$$

Viewing (7) as a quadratic equation in k, we see that $\ell^2m^2+4N\ell^3$ must be a square. Hence

$$m^2+4N\ell = A^2 \qquad \text{......(8)}$$

for a rational integer A. Choose the sign of A in such a way that $2k = (m+A)\ell$. From (8) we deduce that

$$(A-m)(A+m) = 4N\ell$$

and hence there are rational integers u and v such that

$$A-m = 2u \ , \ A+m = 2v \ , \ uv = N\ell .$$

Expressing k and m in terms of u,v and ℓ , thus $k = v\ell$ and $m = v-u$, and inserting these expressions in (6) yields equation (4). The converse is obvious. Note that for each pair (u,v) there is at most one $\ell \neq 0$ such that (3) and (4) are satisfied. This means that each pair (u,v) with $uv \neq 0$ and satisfying (3) and (4), determines a solution (x,y) of (1) uniquely. □

COROLLARY *A solution* (x,y) *of* (1) *with* $N > 0$ *is proper and non-trivial iff there exist rational integers* u,v *and* $\ell \neq 0$ *satisfying* (3) *and* (4) *and* $u \geq 2$, $v \geq 1$. *There are two exceptions to this rule. They are*

i) $N = 2$, $(x,y) = (-3,-1)$ *with* $u = 1$, $v = -4$, $\ell = -2$

ii) $N = 3$, $(x,y) = (-2,-1)$ *with* $u = 1$, $v = -3$, $\ell = -1$.

Proof. From (4) it follows that $(u-1)(v+1) \geq 0$. This gives the four possibilities:

1) $u \geq 2$, $v \geq 1$; 2) $u = 1$, v arbitrary; 3) u arbitrary, $v = -1$; 4) $u \leq -1$, $v \leq -2$. Note that $u = 0$ or $v = 0$ gives the trivial solution.

If $u = 1$, then $v = \ell$ or $v = \ell-2$. In the former case we find that $y = 0$. Hence $v = \ell-2$. Then $N \neq 1$ and $v = -2N/(N-1)$. This gives $N = 2$, $v = -4$ or $N = 3$, $v = -3$.

If $v = -1$, then $u = \ell$ or $u = \ell+2$. Now $u \neq \ell$, for $u = \ell$ implies $N = -1$. From $u = \ell+2$ it follows that $u = 2N/(N+1)$ and hence $N = 1$, $u = 1$. But this gives $y = 0$. In the remaining cases $uv \geqq 2$. Rewriting (4), we obtain

$$\ell^2 - 2\ell(u+v) = 1-(u-v-2)^2 \qquad \ldots\ldots(9)$$

Now if $u-v = 2$, then $\ell(\ell-2(u+v)) = 1$ and hence $\ell = 1$ and $\ell-2(u+v) = 1$. However, $u+v \neq 0$. Consequently, $\ell^2 \leqq 2\ell(u+v)$ which shows that $u+v > 0$. □

<u>EXAMPLE</u> We solve the title equation for $N = 1$.

Let (x,y) be a proper non-trivial solution. According to the COROLLARY there are rational integers $u \geqq 2$, $v \geqq 1$ such that

$$(u+v-uv)^2 = 4(u-1)(v+1)+1.$$

Thus $(u-1)(v-1)(uv-u-v-1) = 4(u-1)(v+1)$.

Since $u \neq 1$, we divide by $u-1$. This yields

$$(v-1)(uv-u-v-5) = 8.$$

Clearly, $v-1 = 2^\alpha$ for some $\alpha \in \{0,1,2,3\}$. We find

$v = 2$, $u = 15$, $\ell = 30$ and $(x,y) = (9,-21)$,

$v = 3$, $u = 6$, $\ell = 18$ and $(x,y) = (8,-10)$ and

$v = 5$, $u = 3$, $\ell = 15$ and $(x,y) = (9,-6)$.

There are no other proper non-trivial solutions.

<u>LEMMA 2</u> *If* $N > 1$ *and* $u \geqq 2$, $v \geqq 1$ *satisfy* (3) *and* (4) *then*

$$\min(u,v) < 4N$$

and

$$\max(u,v) < \tfrac{2}{3}N^3 \quad , \textit{provided } N \neq 2 \textit{ or } 4.$$

Proof. Recall (9). According to the COROLLARY, $u-v \neq 2$. If $|u-v-2| = 1$, then $\ell = 2(u+v)$ since $\ell \neq 0$. Suppose

$u-v = 1$. Then $4u = \ell+2$ and $4v = \ell-2$. From (3) it follows that $\ell^2-16N\ell-4 = 0$. It is easy to see that this is impossible. If $u-v = 3$, then $4u = \ell+6$ and $4v = \ell-6$. This yields $\ell^2-16N\ell-36 = 0$, which is also impossible because $N > 1$. Hence $(u-v-2)^2 \geqq 4$. Then (9) implies $\ell^2-2\ell(u+v) < 0$ and thus $uv/N = \ell < 2(u+v)$. This gives $uv/(u+v) < 2N$ and this shows that $\min(u,v) < 4N$.

If we eliminate ℓ from (3) and (4), we find the equivalent expressions

$$u^2(v-N)^2-2uN(v^2+vN+2N)+N^2(v+1)(v+3) = 0 \qquad \ldots\ldots(10)$$

and

$$v^2(u-N)^2-2vN(u^2+uN-2N)+N^2(u-1)(u-3) = 0 \qquad \ldots\ldots(11)$$

If $v = N$ then it follows from (10) that $4u = N+3$. Similarly, if $u = N$ then (11) implies $4v = N-3$. Now suppose that $v \neq N$ and $u \neq N$. We shall show that $|u-N| \geqq 3$ and $|v-N| \geqq 3$, provided that $N \geqq 6$.

From (10) we deduce that $u(v-N)^2/N \in \mathbb{N}$. For, if $u/N = p/q \in \mathbb{Q}$ with $(p,q) = 1$, then q divides $(v-N)^2$. Put $\delta_1 := u(v-N)^2/N \in \mathbb{N}$ and $\delta_2 := 2(v^2+vN+2N)-\delta_1$. Then $\delta_1\delta_2 = (v-N)^2(v+1)(v+3) \in \mathbb{N}$ and hence $\delta_2 \in \mathbb{N}$. Now

$$[\delta_1-\tfrac{1}{2}(v-N)^2]\cdot[\delta_2-\tfrac{1}{2}(v-N)^2] = (v-N)^2[v(4-N)+3-2N+\tfrac{1}{4}(v-N)^2]$$

and

$$[\delta_1-\tfrac{1}{4}(v-N)^2]\cdot[\delta_2-\tfrac{1}{4}(v-N)^2] = (v-N)^2[(\tfrac{1}{2}v+1)(v-N+2)+2v+1+(v-N)^2/16].$$

Consequently, if $0 < |v-N| \leqq 2$ and $N \geqq 4$ then

$$[\delta_1-\tfrac{1}{2}(v-N)^2]\cdot[\delta_2-\tfrac{1}{2}(v-N)^2] < 0$$

and

$$[\delta_1-\tfrac{1}{4}(v-N)^2]\cdot[\delta_2-\tfrac{1}{4}(v-N)^2] > 0 .$$

Thus $\frac{1}{4}(v-N)^2 < \delta_i < \frac{1}{2}(v-N)^2$ for $i = 1$ or 2. This is clearly impossible. Analogously, consider (11) and put $\varepsilon_1 := v(u-N)^2/N \in \mathbb{N}$ and $\varepsilon_2 := 2(u^2+uN-2N)-\varepsilon_1$. Then $\varepsilon_1\varepsilon_2 = (u-N)^2(u-1)(u-3)$. If $u = 2$ then $(N-2)^2\ell^2 = 16\ell+4$ and this gives $N = 5$. Assume $N \neq 5$, then $u \geqq 3$ and thus $\varepsilon_1\varepsilon_2 \in \mathbb{N}\cup\{0\}$. Again

$$[\varepsilon_1-\tfrac{1}{4}(u-N)^2]\cdot[\varepsilon_2-\tfrac{1}{4}(u-N)^2] = (u-N)^2[(\tfrac{1}{2}u-1)(u-N-2)-2u+1+(u-N)^2/16].$$

If $0 < |u-N| \leqq 2$ then $[\varepsilon_1-\frac{1}{4}(u-N)^2]\cdot[\varepsilon_2-\frac{1}{4}(u-N)^2] < 0$ and hence $0 \leqq \varepsilon_i < \frac{1}{4}(u-N)^2 \leqq 1$ for $i = 1$ or 2. This means that $\varepsilon_1\varepsilon_2 = 0$, which implies $u = 3$. But then $(N-3)^2\ell = 6(N+9)$ and $|N-3| = 1$ or 2. Hence $N = 4$ or 5. We have shown that $|v-N| \geqq 3$ and $|u-N| \geqq 3$ if $N \geqq 6$.

From (10) we deduce

$$u < 2N(v^2+vN+2N)/(v-N)^2 \leqq \tfrac{2}{9}N[(N+3)^2+N(N+3)+2N],$$

because the function $(x-N)^{-2}(x^2+xN+2N)$ of x is increasing on $(0,N)$ and decreasing on (N,∞). It is not difficult to check that $N \geqq 12$ now implies that $u < \frac{2}{3}N^3$. Similarly, from (11) we obtain, recalling that $u \geqq 3$ if $N \neq 5$

$$v \leqq 2N(u^2+uN-2N)/(u-N)^2 \leqq \tfrac{2}{9}N[(N+3)^2+N(N+3)-2N]$$

and this gives also $v < \frac{2}{3}N^3$ if $N \geqq 9$. It remains to check that $\max(u,v) < \frac{2}{3}N^3$ in the cases $N = 2,3,\ldots,11$. We find that $N = 2$ and $N = 4$ are the only exceptions. □

3. THE PROOF OF THEOREM 1

If we suppose that $N > 2$, $N \neq 4$, $u \geqq 2$ and $v \geqq 1$, then $\ell \leqq 2(u+v) = 2(\min(u,v)+\max(u,v)) < 2(\frac{2}{3}N^3+4N)$, according to LEMMA 2. It is easy to see that

$$(2x+1)^2 = 4\ell(v+1)+1 \text{ and } (2y+1)^2 = 4\ell(u-1)+1.$$

Hence

$$\max((2x+1)^2,(2y+1)^2) < 8(\tfrac{2}{3}N^3+4N)(\tfrac{2}{3}N^3+1)+1 < 8(\tfrac{2}{3}N^3+4N)^2.$$

This gives $\max(|x|,|y|) < N^3$, provided $N \geqq 10$. The only exceptions are found to be $N = 1,2$ and 4. This completes the proof of THEOREM 1.

4. THE CONSTRUCTION OF SOLUTIONS

It is an easy consequence of LEMMA 2 that the total number of solutions of equation (1) for a given $N > 0$ is bounded from above by $8N$. In the light of our computations (see the table) this does not seem to be a very realistic upper-bound. To see what is involved, we consider (10). Suppose $v \neq N$ and consider (10) as a quadratic equation in u. Then there must be a rational integer z such that

$$(v^2+vN+2N)^2-(v-N)^2(v^2+4v+3) = z^2 \qquad \text{......(12)}$$

and

$$u = N(v^2+vN+2N\pm z)/(v-N)^2 \qquad \text{......(13)}$$

Now (12) may be written as

$$z^2 = 4(N-1)(v+1)^3+(N-3v-2)^2 \qquad ......(14)$$

and it is not difficult to see that this is an equation for an elliptic curve in the (v,z)-plane, provided $N > 1$. It is well-known however, that finding points with integral co-ordinates on a given elliptic curve - or rather on a Weierstrass model with integral coefficients of the curve - is very hard indeed.

In the construction of solutions of equation (1) we may use THEOREM 1 i.e. we may search for all x and y with $\max(|x|,|y|) < N^3$. However, it is easier and more effective to use (13) and (14) and the equivalent relations we obtain by setting off from (11) instead of (10):

$$v = N(u^2+uN-2N\pm w)/(u-N)^2 \qquad ......(13')$$

and

$$w^2 = 4(N+1)(u-1)^3+(N-3u+2)^2 \qquad ......(14')$$

Indeed, we consider all those values of v with $1 \leqq v < 4N$, $v \neq N$ for which there is a z satisfying (14). For each such combination we decide whether u, given by (13) for one or the other sign is integral. And then we do likewise using (13') and (14'), where v is replaced by u and z is replaced by w. This is the way in which we constructed the table.

It is easy to see that there are infinitely many $N > 1$ for which equation (1) admits of at least one non-trivial solution. In this context we have the following theorem.

THEOREM 2 *Suppose that* $N > 1$. *Then*

(i) if N *is odd, there is at least one non-trivial solution of* (1), *and*

(ii) if $27N^2-2$ *is a square, there are at least five non-trivial solutions of* (1).

Proof. (i) If $N \equiv 1 \pmod 4$, then $u = \frac{1}{4}(N+3)$, $v = N$ gives the solution $(x,y) = (\frac{1}{2}(N+1), \frac{1}{4}(N-1))$ and if $N \equiv 3 \pmod 4$, we find from $u = N$, $v = \frac{1}{4}(N-3)$ the solution $(x,y) = (-\frac{1}{4}(N+1), -\frac{1}{2}(N-1))$.

(ii) From (13), respectively (13') we observe that more than one non-trivial solution may be expected if both signs give rise to an integral u, respectively v. It is not very difficult to prove that under that assumption $4N/(v-N) \in \mathbb{Z}$, respectively $4N/(u-N) \in \mathbb{Z}$.

Now if $u = 3N$, we find from (3) and (4) that

$$(2v-6N+1)^2 = 27N^2-2$$

and similarly, if $v = 3N$ we obtain

$$(2u-6N-1)^2 = 27N^2-2.$$

Thus, if there is an $r \in \mathbb{N}$ such that $27N^2-2 = r^2$, then $u = 3N$ gives $2v = 6N-1\pm r$ and for both signs $v \neq 3N$. Further $v = 3N$ gives $2u = 6N+1\pm r$. This leads to the four non-trivial solutions

$$(x,y) = (\tfrac{1}{2}(9N-1\pm 2r), \tfrac{1}{2}(-9N+2\mp r)),\ (\tfrac{1}{2}(9N+2\pm r), \tfrac{1}{2}(-9N-1\mp 2r)).$$

Keeping (i) in mind and checking that no overlap occurs, we find that there are at least five non-trivial solutions. □

Note that the Diophantine equation

$$27N^2-2 = r^2 \qquad \ldots\ldots(15)$$

has infinitely many solutions in positive rational integers N and r. Therefore we may deduce from THEOREM 2 that there are infinitely many values of N with $N > 1$ for which equation (1) has at least six solutions. The smallest $N > 1$ which satisfies (15) for some $r \in \mathbb{N}$ is $N = 51$. Equation (1) with $N = 51$ has precisely seven solutions.

TABLE

Proper non-trivial solutions (x,y) of equation (1) with N > 0. No entry indicates that no such solutions exist.

N	(x,y)
1	(8,-10), (9,-6), (9,-21)
2	(-3,-1), (15,-25), (54,-12)
3	(-2,-1)
4	(90,12)
5	(-6,-16), (3,1), (27,6)
6	(14,4), (64,-40)
7	(-2,-3), (9,3), (50,-120)
9	(5,2)
11	(-3,-5)
13	(7,3)
15	(-9,-21), (-4,-7), (2,1), (104,-169)
17	(9,4)
18	(-25,-85), (4,2), (207,-1587), (209,-121)
19	(-5,-9)
21	(11,5)
22	(169,39)
23	(-6,-11), (867,-187)
25	(13,6)
27	(-7,-13)
29	(15,7), (125,35)
31	(-8,-15)
32	(539,-217)
33	(17,8)
34	(-70,-300)
35	(-9,-17)

TABLE (continued)

N	(x,y)
37	(19,9)
38	(2883,279)
39	(-10,-19)
40	(-4,-6), (441,-273)
41	(21,10)
42	(50,20), (289,-697)
43	(-11,-21)
45	(23,11)
46	(329,-441)
47	(-12,-23)
49	(5,3), (25,12)
50	(351,-507)
51	(-247,-1805), (-36,-96), (-13,-25), (98,35), (363,-495), (494,-361)

Equations which admit of only the trivial solution in the range $1 \leqq N \leqq 51$ are those with N = 8, 10, 12, 14, 16, 20, 24, 26, 28, 30, 36, 44 and 48.

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