ARTICLES

On the Shape of a Violin

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My mother was a professional musician, and I loved listening to her playing the piano for hours on end. We, the children, too had to choose an instrument for our musical education, and at the age of eight, I began to study the cello. It took a while before I got the hang of it, but the practicing paid off and I began to really enjoy playing. I also liked being invited to join the school orchestra and several chamber music groups, and I became aware of the emotional impact the sound of a dead piece of wood could have on people. I promised myself that, given the chance, I would learn more about these miraculous stringed instruments and how they are made.

I became a mathematician, and I spend the next 40 years teaching and thinking about mathematics. Then after I retired, my sister and I happened to pass by the shop of a violin maker—a "luthier" to use the right word. She—the luthier—was very helpful, and I went on to a serious course in instrument making, building classical instruments in the tradition of the great masters of the past. I recently finished a viola da gamba, but I still have a long way to go in unraveling the secret of its magic sound.

This article is about one way in which mathematics can be applied to the art of lutherie. It addresses this particular problem: How can we give a mathematical description of the backplate of the violin? We limit ourselves to its first appearance as a two-dimensional piece of wood.

The overall shape of instruments of the violin family (violin, viola, cello) has changed very little over the past hundreds of years, in contrast to that of most other stringed instruments. So, this shape can certainly be seen as rather successful, possibly even ideal in the sense of most natural or most visually pleasing. We will use the mathematics of cubic splines, in particular parametrized cubic splines. The main reason for choosing cubic splines lies in their unique curvature properties, which might help us in our attempt to give an explanation for this ideal shape. These techniques may become very helpful to luthiers. But our main purpose here is to describe them to mathematicians and to demonstrate their strengths in the context of this wonderful application.

The backplate

Studying the ways in which stringed instruments are constructed, I learned that, on the one hand, it is common practice to copy famous instruments in minute detail, on the

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other hand, if a new model is desired, with a few exceptions, only marked ruler and compass are used in the construction. Often, the ways in which such constructions are laid out are complicated and appear to be rather ad hoc (see [**1**, Tafel I]). “We need a round curve here, so which circle serves our purpose best,” seems to be the adage, and where circles meet, the sharp intersection is generally smoothed over. An exception is the catenary, a curve defined by the formula

$$y = a\cosh(x/a) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \text{ with } a > 0,$$

which sometimes helps to shape the arching of the plates of the violin. The constant a can be expressed in terms of the length of the centerline of the backplate. But generally mathematical formulae are shunned.

The measurements required for the construction sometimes follow certain patterns, like those based on the golden section (see [**5**]). Often, they seem to come from local considerations and follow no general rule or philosophy.

Figure 1 Template (left) and backplate (right) of base model after Antonio Stradivari 1689.

When starting on a new violin, it is common practice to first make a template that serves as a model for the mould to which the sides of the violin (the ribs) have to be glued (see FIGURE 1, left). The ribs are approximately 1.2 mm thick, and both front and back plates protrude from the sides by another 2.5 mm so that the outer form of the violin is slightly larger, which is especially noticeable at the points where the upper and lower parts change into the Cs (see FIGURE 1, right). Here, we are looking at the two-dimensional form of the soundbox of a violin. At this point, I would like to draw your attention to the wonderful treatise on the *Art of Violin Making* by Chris Johnson and Roy Courtnall [**8**].

When designing a new violin, it is the template that we have to construct first. However, since we wish to compare our construction to actual instruments, we shall instead focus on the outer form. We choose the backplate as the best representative of this form, and so we ignore the neck and the scroll. Moreover, the measurements we shall use are those of a Stradivari violin built in 1689; from here on, we shall refer to this instrument as the *base model**; see FIGURE 1. A list of 115 points from which the outer form of the backplate of this model can be constructed is available at [**16**].

Observe that the backplate is symmetrical, so it is enough to describe the curve on one side. The piecewise smooth curve in FIGURE 2 consists of five pieces, the lower curve $L_1 \dots L_6$, the upper curve $U_1 \dots U_6$, the C-curve $C_1 \dots C_7$, and two short line segments connecting the lower curve with the C-curve and the upper curve with the C-curve, L_6C_1 and U_1C_7. All these curves lie in the same plane. The positions of the L, C, and U points will be clarified in due course.

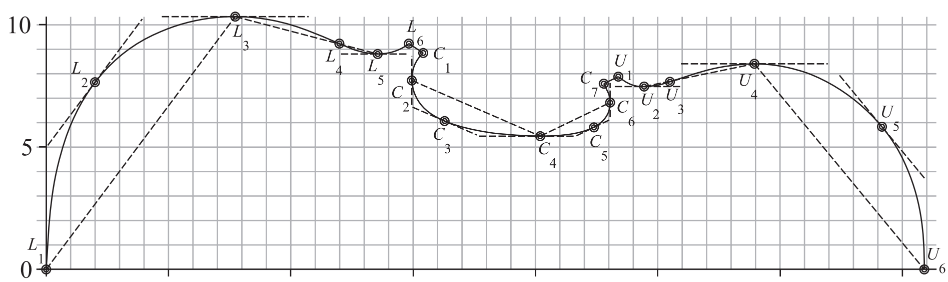


Figure 2 Significant points on the base model's outer form with construction lines.

It is obvious that starting the construction we have to set off with a number of given measurements. These roughly determine the outer form of the instrument. Initially, we have to decide on the length of the center line, which divides the back in two symmetric halves. It is such a symmetric half we are interested in. We also have to know the position and size of the C-parts and the largest and smallest width on the upper and lower parts. All this is necessary so that the final result could rightly be called a violin. This also means that, although we may choose our own measurements, we must not loose sight of the fact that the margins are rather small. Typical measurements of a standard violin are given in TABLE 1.

TABLE 1: Standard backplate measurements.

Standard violin	mm
length	356
upper width	168
middle width	112
lower width	208

The measurements vary only slightly with a variation of the length of the body of at most 10 mm. Useful information on measurements is given in [**15**].

Next, we choose a number of points on the outline of the backplate in agreement with the measurements we set out with. We shall refer to these points as *guide points*. There are a few obvious ones, like the endpoints of the center line U_6 and L_1 and the other endpoints of upper and lower parts (U_1, L_6) and of the C-part (C_1, C_7) and maybe a few others, corresponding to maximal width (U_4, L_3) and minimal width (C_4) for instance (see FIGURE 2). We shall continue the discussion on the choice of guide points, after first looking at cubic splines.

*This model is in use in the violin class at the CMB (Centrum voor MuziekinstrumentenBouw); see [**4**].

Cubic splines, the basics

Now let us state our mathematical problem.

We are given a sequence of distinct guide points (x_i, y_i), $i = 0, 1, \ldots, n$, in the plane, and we are seeking to find a "nice" plane curve passing through these points in the order given by $i = 0, 1, \ldots n$. Here, "nice" means smooth, visually attractive, and with no unnecessary bending.

Let us first assume these guide points are placed in such a way that the graph of a proper function s can pass through them. Later on, we will extend this to the case of points that are placed arbitrarily. Then, because the graph of a function can obviously not turn on itself, the x-values (the *knots*) must be ordered like $a := x_0 < x_1 < \cdots < x_n =: b$. So $y_i = s(x_i)$ for each i. Clearly, this is an interpolation problem, so let us consider polynomial interpolation. Because high degree polynomial interpolation usually comes with many oscillations, we should go for piecewise low degree polynomial interpolation. Piecewise linear interpolation is not smooth at the knots, and piecewise quadratic polynomial interpolation does not give us enough freedom to control the smoothness at the knots. Therefore, we choose piecewise cubic interpolation.

The function $s : [a, b] \to \mathbb{R}$ is called a *cubic spline* if it satisfies the following conditions:

1. $s = s_i$ is a cubic polynomial on the subinterval $[x_{i-1}, x_i]$ of [a,b] for $i = 1, \ldots, n$,
2. $s(x_i) = y_i$ for $i = 0, \ldots, n$,
3. $s_i^{(j)}(x_i) = s_{i+1}^{(j)}(x_i)$ for $i = 1, \ldots, n-1$ and $j = 0, 1, 2$.

The first condition tells us that s is a piecewise cubic polynomial on $[a, b]$, and the second one says that s is an interpolation function on the set of guide points. The third condition expresses the smoothness of s at the knots. It says $s \in C^2[a, b]$, which means that s is a twice continuously differentiable function on the closed interval $[a, b]$. The s_i are cubic polynomials, and therefore, s can be explicitly given by $4n$ coefficients. On the other hand, the interpolation and smoothness conditions amount to a total of $n + 1 + 3(n - 1) = 4n - 2$ equations. Therefore, we may impose two extra boundary conditions. A natural choice is $s''(a) = s''(b) = 0$, which gives the so-called *natural cubic spline*. Another choice is to fix the vector $(s'(a), s'(b))$, and this is known as the *clamped cubic spline*. The cubic spline function s is uniquely determined by the three conditions plus the two boundary conditions. In the next section, we shall give a proof of this by construction.

Now, the curves we need for our purpose cannot always be given by the graphs of spline *functions*. This can clearly be seen in FIGURE 2: The C-curve, when run through from left to right, turns on itself, and is therefore not the graph of a function. That is why we need so-called *parametric splines*. We understand a parametric spline to be a plane curve given by the set of points

$$\{(x(t), y(t)) : a \le t \le b\},$$

where x and y are spline functions of the parameter t. Often, one chooses $a = 0$ and $b = 1$. How should the parametrization be chosen? This is an important point. The interpolation points are generally not uniformly spaced, and therefore, different parametrizations should give different splines. Also, the curves we are after are certainly nonsingular, so our parametrized curves should also be nonsingular. The most natural choice is to take t to be the arc length of the curve. But it is almost always rather difficult to find an explicit expression for the arc length of a given curve, and our curves are no exception. We should also take into account the fact that successive guide points are not placed at equal distances. So, if for $i = 0, 1, \ldots, n$ the points

$A_i = (x_i, y_i)$ are the guide points, then we define $l_0 := 0$, $l_{i+1} := l_i + \|A_{i+1} - A_i\|_2$ for $i = 0, 1, \ldots, n-1$ so that l_n is the sum of the line segments joining successive guide points, which is then the piecewise linear approximation of the arc length between A_0 and A_n. Then choose $t_i := l_i/l_n$ so that $t_0 = 0$ and $t_n = 1$. Thus, $t_0, t_1, \ldots, t_n$ are the knots of the spline functions $x(t)$ and $y(t)$, the components of the parametric spline $(x(t), y(t))$ with $0 \le t \le 1$. Further, $x_i = x(t_i)$ and $y_i = y(t_i)$ for all i.

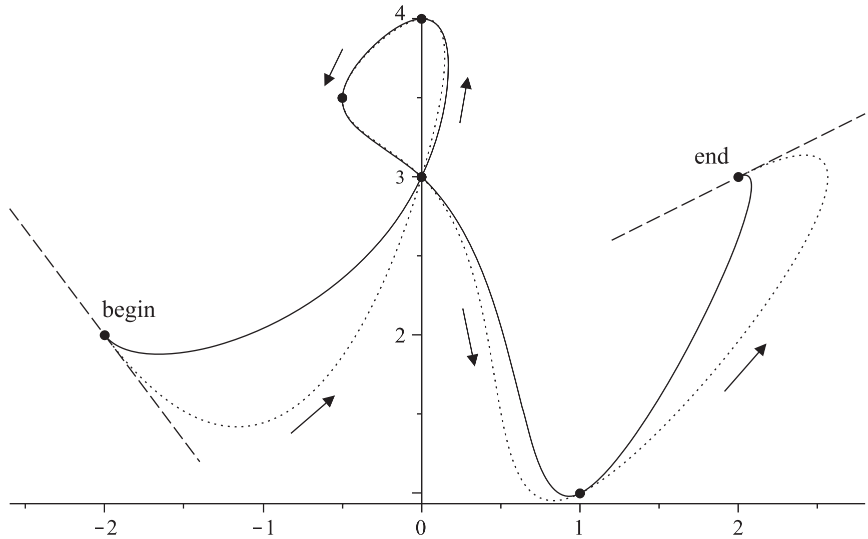


Figure 3 Parametric cubic splines through the same points and with equal end conditions but with different multiplication factors at the endpoints.

Dealing with clamped cubic spline functions, we need to choose the tangent directions at both endpoints to make the splines unique. With parametric clamped cubic splines, we have even more freedom. Indeed, let the curve be given by $f(x, y) = 0$ with a parametrization as given above. Traversing the curve from $t = t_0$, let us set off in the direction of the vector α with $\|\alpha\|_2 = 1$. This means $(x'(t_0), y'(t_0)) = \alpha$ because $[x'(t_0), y'(t_0)]$ is the direction of the tangent to the curve at A_0. However, this fixes neither $x'(t_0)$ nor $y'(t_0)$—required for clamped cubic splines—because $m \cdot \alpha$ gives the same direction for all real $m \neq 0$. This can also be seen as follows. As the curve is nonsingular, either $x'(t_0) \neq 0$ or $y'(t_0) \neq 0$. Without loss of generality, we assume $x'(t_0) \neq 0$. It then follows, that in a neighborhood of A_0, the curve can be given by $y = \phi(x)$ for a differentiable function ϕ. Now let $x'(t_0) = m \cdot \alpha_1$ and $y'(t_0) = m \cdot \alpha_2$, then $\phi'(x(t_0)) = y'(t_0)/x'(t_0) = \alpha_2/\alpha_1$, and the factor m drops out. We shall call this factor m the *multiplication factor*. So the value we choose for m does not affect the direction of the tangent at A_0. Changing the multiplication factor does not alter the tangent, but the larger m, the closer the graph of the curve is drawn toward the tangent. Naturally, this also applies to the other endpoint A_n. We shall always take the multiplication factor positive. We therefore may have to alter the sign of the direction of the tangent, depending on the way we traverse the curve. In FIGURE 3, the multiplication factor of the dotted spline is much larger than that of the black spline. The directions of the tangents (the dashed lines) at the endpoints are $[3, -4]$ and $[-2, -1]$, respectively.

Cubic splines, advanced properties

As we saw in the previous section, a cubic spline determines $4n - 2$ equations in $4n$ unknown coefficients. The existence and uniqueness of natural and clamped cubic splines can be shown by construction.

The function s'' is a piecewise linear polynomial. Write $\sigma_i = s''(x_i)$ for $i = 0, 1, \ldots, n$. Then for $i = 1, \ldots, n-1$ we have

$$s''(x) = \begin{cases} \frac{x-x_i}{x_{i+1}-x_i}\sigma_{i+1} + \frac{x_{i+1}-x}{x_{i+1}-x_i}\sigma_i & \text{for } x_i \le x \le x_{i+1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}\sigma_i + \frac{x_i-x}{x_i-x_{i-1}}\sigma_{i-1} & \text{for } x_{i-1} \le x \le x_i \end{cases},$$

because s'' is a linear function. Working out the repeated integrals

$$\int_{x_i}^{x_{i+1}} \int_{x_i}^{x} s''(t)\,dt\,dx \quad \text{and} \quad \int_{x_{i-1}}^{x_i} \int_{x}^{x_i} s''(t)\,dt\,dx$$

in two ways and eliminating $s'(x_i)$ from the resulting equations eventually leads to the relations

$$\omega_i\sigma_{i-1} + 2\sigma_i + (1-\omega_i)\sigma_{i+1} = r_i \quad \text{for } i = 1, \ldots, n-1, \tag{1}$$

where

$$\omega_i := \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad \text{and} \quad r_i := \frac{6}{x_{i+1} - x_{i-1}}\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right).$$

Combining relations (1) into a single matrix equation, for the natural boundary conditions, we get the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \omega_1 & 2 & 1-\omega_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \omega_2 & 2 & 1-\omega_2 & & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 2 & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \omega_{n-1} & 2 & 1-\omega_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} = \begin{pmatrix} 0 \\ r_1 \\ \vdots \\ r_{n-1} \\ 0 \end{pmatrix}. \tag{2}$$

The coefficient matrix of (2) is tridiagonal and strictly diagonally dominant because $0 < \omega_i < 1$ for each i. This means that this system is uniquely solvable (see [**7**, Theorem 6.1.10 on page 349]), which implies that the natural cubic spline exists and is unique. It also gives us the values of $s''(x)$ at the knots. Tracing back down the intermediate relations gives us the coefficients of each s_i. In the case of the clamped cubic spline, we find a slightly more complicated matrix equation, but the conclusions remain the same. For both sets of boundary conditions and for any function $f \in C^m[a,b]$ with $m \ge 2$ for which $y_i = f(x_i)$ for each i, and in addition $f' = s'$ at both endpoints for a clamped cubic spline—which means coinciding tangents at the endpoints—we can show the inequality

$$\int_a^b [f''(x)]^2\,dx \ge \int_a^b [s''(x)]^2\,dx, \tag{3}$$

where either $(s'(a), s'(b)) = \alpha$ or $s''(a) = s''(b) = 0$. The proof runs as follows (see [**11**, Ch. 4]. Consider

$$\int_a^b [f''(x) - s''(x)]^2\, dx + \int_a^b [s''(x)]^2\, dx = \int_a^b [f''(x)]^2\, dx - 2\int_a^b s''(x)[f''(x) - s''(x)]\, dx.$$

Using integration by parts on the last integral on the right gives

$$\int_a^b s''(x)[f''(x) - s''(x)]\, dx = s''(x)[f'(x) - s'(x)]\Big|_a^b - \int_a^b s'''(x)[f'(x) - s'(x)]\, dx.$$

The first term on the right vanishes because of the boundary conditions and so does the second term in view of the fact that s''' is constant on each subinterval (x_{i-1}, x_i). Indeed, if $s'''(x) = c_i$ for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, n$, then

$$\int_a^b s'''(x)[f'(x) - s'(x)]\, dx = \sum_{i=1}^n c_i \int_{x_{i-1}}^{x_i} [f'(x) - s'(x)]\, dx = 0.$$

Inequality (3) has an important geometric interpretation that partly explains the reason for the popularity of the cubic spline. The mathematical *curvature* $\kappa(x)$ of a twice continuously differentiable function $f : [a, b] \to \mathbb{R}$ at the point $x \in [a, b]$ is defined by the formula

$$\kappa(x) = \frac{f''(x)}{(1 + [f'(x)]^2)^{\frac{3}{2}}}. \tag{4}$$

The curvature of a circular arc of radius R is $1/R$ or $-1/R$, depending on the orientation of the circle. Assuming $|f'(x)| \ll 1$ on $[a, b]$—admittedly, this is not always the case—we see that the norm $\| \kappa \|_2^2$ is approximately equal to $\int_a^b [f''(x)]^2\, dx$ so that inequality (3) now says that of all the $C^m[a, b]$ functions with $m \geq 2$ and satisfying the interpolation conditions, *the cubic spline has the smallest total curvature in the sense of the ℓ_2-norm.* Of course, $\int_a^b [f''(x)]^2\, dx$ merely gives a coarse measure of the total curvature.

There is yet another interpretation of inequality (3), and this one explains the reason for the name "spline" that is given to this interpolation function. Consider a thin homogeneous isotropic flexible rod whose center line is given by a function $f : [a, b] \to \mathbb{R}$. Then the total bending energy is given by the formula

$$c\int_a^b \frac{[f''(x)]^2}{(1 + [f'(x)]^2)^3}\, dx \approx c\int_a^b [f''(x)]^2\, dx$$

for a certain constant c and under the assumption $|f'(x)| \ll 1$ on $[a, b]$. If such a rod is forced to go through a number of fixed points, in such a way that only forces perpendicular to the rod are applied, it assumes a position of minimal energy. Therefore, inequality (3) now asserts that the center line of this rod approximately follows the natural cubic spline through these points. Outside of the interval $[a, b]$, no force is applied to the rod, and therefore, it assumes the natural shape of a straight line. In that sense, the boundary conditions $s''(a) = s''(b) = 0$ should be seen as "natural." It now makes sense why this type of interpolation function was given the name "spline" because a mechanical spline is a thin flexible rod that is used by draughtsmen (e.g., in shipbuilding) for drawing smooth curves through a number of fixed points. It was I. J. Schoenberg who introduced the name "spline" in 1946 (see [**12**]). See also the foreword by A. Robin Forrest in [**2**].

Cubic splines are by far the most popular of all spline functions; they are especially useful for approximation purposes. In particular, every function continuous on a closed interval $[a, b]$ can be arbitrarily well approximated on $[a, b]$ by cubic splines provided sufficiently many knots are available. In fact, if s is a cubic spline that interpolates $f \in C^m[a, b]$ at the knots $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\| s - f \|_\infty = O(h^{m+1}) \text{ as } h \downarrow 0, \text{ where } m = 1, 2, 3 \text{ and } h := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

See [**6**] and [**17**, 9.7.3]. General treatises on splines are [**3**] and [**14**].

The construction

Now, we are ready to construct the "ideal" outline of the backplate with as few guide points as possible, where ideal is meant in the sense of "visually pleasing" with no unnecessary bending. Of course, avoiding unnecessary bending goes hand in hand with avoiding the use of too many guide points so that the natural curving property of splines is not thwarted.

Naturally, we want to compare the outline of the base model with the approximation obtained by means of parametric splines with a given small set of guide points, the *spline model*. It turns out that, with the guide points chosen in accordance with the measurements of the base model (all 115 data points), together with the approximate data at the endpoints (see TABLE 2 and the paragraph immediately following FIGURE 4), we get an outline that compares very favorably with that of the actual instrument. From now on, we shall call this outline also the *base model* or the *base contour*.

TABLE 2: Tangent directions with multiplication factors at endpoints in FIGURE 4.

Guide point (endpoint)	Direction tangent	Multiplication factor	
		8-point spline	11-point spline
L_1	[0, 1]	32	34.5
L_6	[1.5, 1]	20	28
C_1	[−1, −1.2]	14.6	13.1
C_7	[−1, 1.8]	13.5	16.2
U_1	[1.2, −1]	17	21
U_6	[0, −1]	24	25

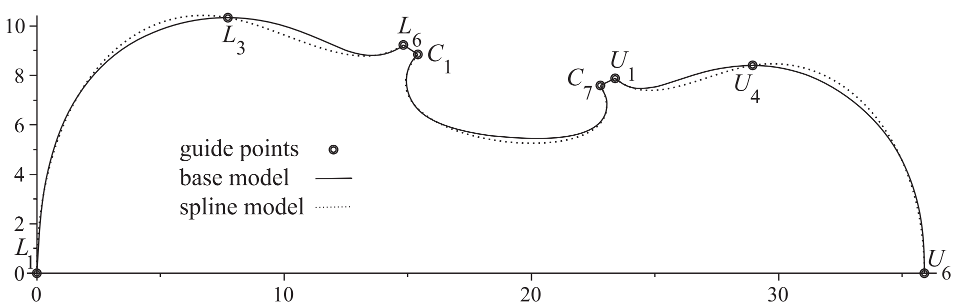


Figure 4 8-point spline compared with the base model.

Choosing the number of guide points and the points themselves is crucial. First, we should not choose too many points for fear of designing the contour ourselves, instead of leaving this to the spline engine. Further, the choice of guide points should be mainly imposed by necessity because of the obvious requirements of size and shape.

The procedure is now as follows. We consider the backplate of our model, or rather half of it, and we try to find points on its contour at the most significant positions. Nineteen possible candidates with an indication on their construction are shown in FIGURE 2. Four groups of points can be distinguished: endpoints (L_1, L_6, C_1, C_7, U_1, U_6), points at extremal positions (L_3, L_5, C_2, C_4, C_6, U_2, U_4), points with tangents parallel to line segments (L_2, C_3, C_5, U_5), and points at the intersection of line segments and the base model (L_4, U_3). From this set of points, we shall choose a subset, the guide points. And finally, these guide points will serve as interpolation points for our parametric cubic splines. We shall also compare the result with the base model.

For each of the parts L, C, U, we shall construct a parametric cubic spline. First, we need to decide on the position of the endpoints of these three parts, namely L_1 and L_6, C_1 and C_7, U_1 and U_6. There cannot be much doubt about the necessity of the choice of these points as guide points, but clearly, this choice cannot possibly be sufficient; at least two more guide points are needed to get anything like the correct shape, typical for a violin. We also must choose the direction of the tangents at these six endpoints and their multiplication factors. An important point is this: We can choose the multiplication factors to suit us best, which could either mean so that we like the resulting curve best or that it matches the base model best. The tangents at L_1 and U_6 must be vertical, and the tangents at the other four endpoints can be freely chosen, say to run at an angle anywhere between 30 to 60 degrees in a positive or negative sense. The upper and lower parts are quite similar, so let us consider the lower part. Provided we choose a point between L_1 and L_6 with an ordinate value larger than that of L_6, the resulting parametric spline through these three points will have the right sort of shape. This is also true for the upper part; the C-part, however, does not need another point. The extra points that certainly satisfy the restrictions are the highest points on the lower and upper parts, namely L_3, and U_4. In FIGURE 4 we compare the shape generated by the cubic parametric splines through these eight guide points with that of the base model. See also TABLE 2 for the tangents and the multiplication factors. The latter are chosen such that the spline curves at the endpoints optimally agree with those of the base model.

To be able to compare our spline model with the base model, we need to know the tangent directions at the endpoints of the base model. Clearly, at the points L_1 and U_6, the exact tangents are known because of symmetry considerations. At the other four endpoints, we do not know the exact tangent directions, but it is not difficult to obtain good approximations. At each of these four points, we have simply taken the direction of the line segment connecting this endpoint with its neighboring point. See TABLE 2 for the result. Next, take point L_1. Choose the multiplication factor by observation in such a way that the lower part of the spline model coincides with the base model as much as possible: Increasing a multiplication factor draws the curve closer toward the tangent. A similar action applies to U_6. The remaining four points have to be treated slightly differently because we do not have the exact tangent directions of the base model at these points. Here, we also have to vary the tangent directions of the base model slightly to obtain a visually optimal situation. Of course, these solutions are not mathematically exact, but that is not the point here. We are mainly interested in the optimal bending properties of cubic splines. Thus, trying to determine the best multiplication factor mathematically is not really important because *any* choice of this factor gives a spline with optimal bending and thus a beautiful model, but here we want to see how well we can cover the base contour!

Although the shape of the spline curve in FIGURE 4 seems quite acceptable, there is also room for improvement. Observe that the points L_3 and U_4 are not really close

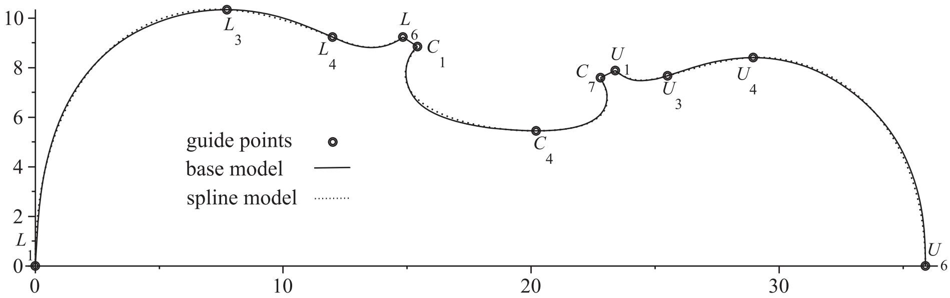


Figure 5 11-point spline compared with the base model.

to extremal positions on the spline and that the C-part spline curve possibly makes the waist too small. So if one wishes to stay closer to the base model, extra points are needed, like L_4, C_4, and U_3. The points L_4 and U_3 have the effect of "flattening off" the splines toward the base model. Of course, any points between L_3 and L_5 and between U_2 and U_4 will have that same effect. Observe that in TABLE 2 the directions of the tangents are the same, but the multiplication factors are not. Although not perfect, the match of the 11-point spline (see FIGURE 5) is quite good. Other choices of guide points can be made, and a perfect match can be easily obtained by taking a few extra points from the 19 significant points of FIGURE 2.

Curvature

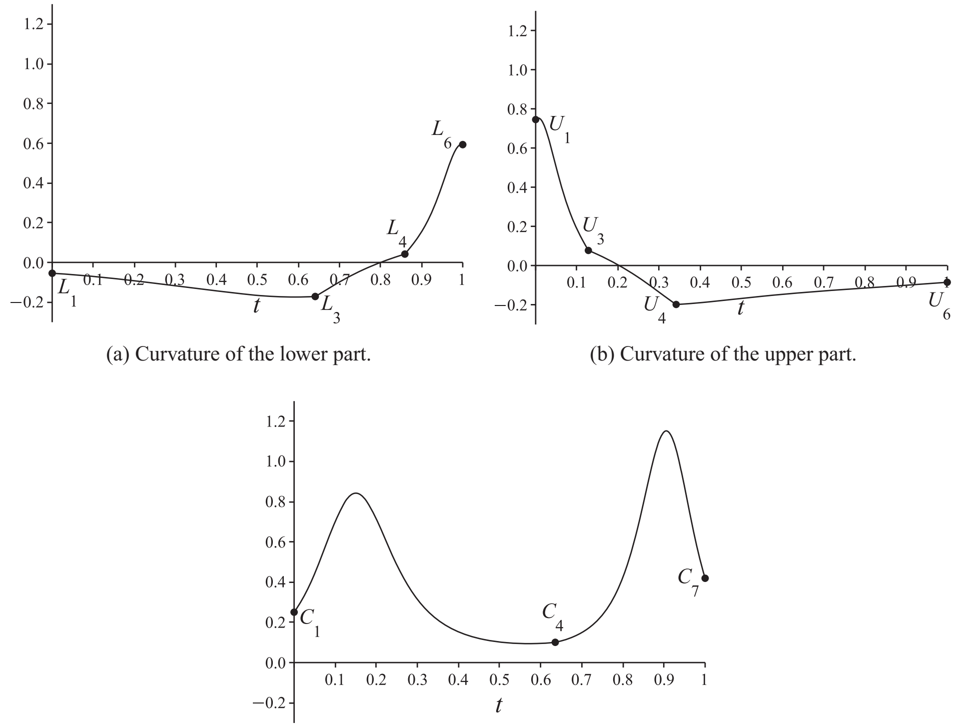
We have seen that the combined spline contours through the 11 given guide points provide a reasonably close fit for the base model. Another interesting point that can be made is about the curvature: How does the curvature of the base model change from point to point? Can the answer throw light on the way luthiers in their marked ruler and compass constructions use circular arcs?

The curvature of a twice continuously differentiable function at each point of its graph is given by equation (4). From this, the formula for the curvature of a parametric curve $C = \{(x(t), y(t)) : a \leq t \leq b\}$ at each value of the parameter t can be deduced easily:

$$\kappa(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{\frac{3}{2}}}. \tag{5}$$

For convenience, we have dropped the t in the right-hand side of formula (5). In order to check this formula, suppose $x'(t) \neq 0$ at the parameter value $t = t_0$. Then the curve C may be given in a neighborhood of t_0 by an equation $y(t) = \phi(x(t))$ for a twice continuously differentiable function ϕ. Then dropping the t again, $y' = \phi'(x)x'$ and $y'' = \phi''(x)(x')^2 + f'(x)x''$, from which in combination with (4) the assertion easily follows. The curves we consider have no singular points. So if $x'(t) = 0$ at $t = t_0$, then certainly $y'(t_0) \neq 0$, and a similar argument may be given. Recall that the circle of curvature at point $t = t_0$ has radius $1/\kappa(t_0)$. It is the circle whose center lies on the normal to the curve at that point and whose curvature agrees with that of the curve at t_0.

It follows from formula (5) that the curvature function $\kappa(t)$ is continuous on the entire range $[t_0, \ldots, t_n]$. Nevertheless, the curvature of C^2-curves is rather sensitive to small changes. If our curve is a parametric cubic spline, it is unlikely that $\kappa(t)$ is differentiable at the knots. So our function $\kappa(t)$ will probably not have a very smooth appearance.



(a) Curvature of the lower part.

(b) Curvature of the upper part.

(c) Curvature of the C part.

Figure 6 Curvature of the parametric 11-point spline curves.

Recall that the Stradivari base model is obtained by careful observation of 115 successive data points about 5 mm apart. Joining these points by means of parametric cubic splines to obtain a close approximation of the model gives a good result, but the corresponding curvature function does not look so good as a result of the phenomenon mentioned above.

Fortunately, the 11-point spline approximation of the base contour is also quite good, and what is more, its curvature function is much smoother. It is therefore preferable to consider the curvature function of our parametric spline approximation. In FIGURE 6, the curvature functions of the lower, C-, and upper parts are shown. It appears that the curvature at both ends L_1 and U_6 is almost constant and small. Although one should be very careful with one's interpretation of this, it could mean that the contour at both L_1 and L_6 is similar to a circular arc. TABLE 3 shows the approximate radius of the curvature circle at each of the guide points. We also observe in FIGURE 6(c) that the curvature of the C-part is rather regular; around the center the curvature is almost constant, which might indicate that the major curve resembles a circle with radius of approximately 9.9 cm. Also, the small circular arcs near the endpoints are clearly visible.

Note on the practical application

So far we have considered the mathematical definition and properties of parametric cubic splines. But how can these be handled in practice? All calculations done so far, including all graphs, have been created by means of the computer algebra package Maple (see [**9**]). But in order to use splines, one does not need intimate knowledge

TABLE 3: Coordinates (x, y) with parameter value $t \in [0, 1]$ of the 11 guide points (see FIGURE 5) with their curvature $\kappa(t)$ and radius R of the curvature circle. Coordinates and radius are measured in centimeters

Guide point	$x(t)$	$y(t)$	t	$\kappa(t)$	R
L_1	0.0	0.0	0	-0.0538253196	18.57861703
L_3	7.7279227	10.3352606	0.6406923777	-0.1707587319	5.85621590
L_4	11.9861762	9.2333787	0.8590647440	0.0425922197	23.47846643
L_6	14.8249271	9.2336330	1	0.5932583552	1.68560627
C_1	15.4128687	8.8493857	0	0.2506912160	3.98897104
C_4	20.1975232	5.4481232	0.6353508310	0.1009567893	9.90522784
C_7	22.7944348	7.5946695	1	0.4193279880	2.38476808
U_1	23.3900054	7.8817742	0	0.7454555014	1.34146169
U_3	25.4994239	7.6701966	0.1280534854	0.0777210168	12.86653264
U_4	28.9525636	8.4071580	0.3413289745	-0.1978771462	5.05364070
U_6	35.8974966	0.0	1	-0.0845792332	11.82323322

of a computational nature. It is sufficient to know how to work with a CAD system, like VectorWorks, TurboCad, or AutoCAD, to name but a few. These systems use vector graphics and they implement parametric cubic splines through B-splines (B stands for basis). These systems are mainly used for experimental design purposes, where curves are designed and used interactively. Now, cubic splines—at least in the way presented here—are less useful for experimental purposes because changing even a single interpolation point changes the entire spline instead of the parts closest to the point changing so that all calculations have to be done all over again. B-splines cure that problem by constructing the spline locally between two successive interpolation points as a linear combination of (four, in case of cubic splines) basis splines. Further, in design purposes, there are generally no severe restrictions as to the points the curve has to pass through. With B-splines one uses so-called *control points* to change the shape of the curve, but the curve generally does not pass through these control points. So, mathematical splines are calculated by interpolation, and B-splines use a different computational technique based on the linear combination of basis splines. For those who wish to know more about the mathematical background of B-splines, Bézier curves and NURBS (nonuniform rational B-splines) we refer to [**13**] for a very nice and detailed overview. See also [**10**].

Conclusion

We have seen that by choosing guide points in a certain way a quite acceptable model for the backplate of a violin can be constructed with the nice curvature properties of cubic splines. Of course, the great masters would have had no need for these fancy methods—if they had known them—in their quest for the most suitable but also most gracious and beautiful shape, but we, ordinary mortals, might find the help modern methods can give us quite useful. These modern methods are accessible through CAD software. Even so, and fortunately too, in the search for suitable guide points, human intervention—in other words, the eye of the master—remains indispensable.

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Summary. For centuries luthiers—that is, instrument makers of violins and other stringed instruments—had no more sophisticated tools at their disposal to define the shape of their instruments than marked ruler and compass. Today, modern aids are available in terms of computational power and expertise in graphic design to assist them in this respect. This raises the following question: How can these powerful computational techniques be applied in the process of searching for a form of the violin both pleasing to the eye and optimal in some mathematical sense? In this paper, I use parametric cubic splines in an attempt to come close to and possibly improve upon—strictly in a mathematical and visual sense—the shape of a violin as laid down by the great masters of the past. The main reasons for choosing the cubic spline are: good approximation properties, simplicity of construction, and most importantly, its unique curvature properties.

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