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APPROXIMATIONS OF THE EIGENVALUES OF THE COVARIANCE MATRIX OF A FIRST-ORDER AUTOREGRESSIVE PROCESS

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This paper provides explicit estimates of the eigenvalues of the covariance matrix of an autoregressive process of order one. Also explicit error bounds are established in closed form. Typically, such an error bound is given by $\varepsilon_k=(4/(n+1))^{\frac{1}{2}}\rho^2\sin(k\pi/(n+1))$, so that the approximations improve as the size of the matrix increases. In other words, the accuracy of the approximations increases as direct computations become more costly.

1. Introduction

Consider the autoregressive process of order one,

$$Y_t=\rho Y_{t-1}+e_t, \qquad t\in \boldsymbol{Z},$$

where ρ is real with $|\rho|<1$, e_t is stochastically uncorrelated with the other e's and with all Y_s for $s<t$. Moreover, e_t has zero mean and variance σ^2. The time series Y_t converges as $t\to\infty$ to a stationary time series, because $|\rho|<1$. The $n\times n$ covariance matrix of n observations has the form $\sigma^2(1-\rho^2)^{-1}\Gamma$, where $\Gamma=\Gamma_{n,\rho}$ is the matrix with kth row vector

$$(\rho^{k-1},\rho^{k-2},\ldots,\rho,1,\rho,\ldots,\rho^{n-k-1},\rho^{n-k}), \qquad k=1,\ldots,n.$$

Sometimes, e.g. in connection with power comparisons of a test of the hypothesis $\rho=\rho_0$ [cf. Dickey and Fuller (1979)] or in the spectral theory of time series [cf. Fuller (1976, ch. 4)], it might be useful to have information on the eigenvalues of Γ in explicit form. However, there appears to be no explicit form for the eigenvalues [see Grenander and Szegö (1958, p. 70)], except for the trivial case $\rho=0$, which we shall exclude from now on. In this paper we intend to give certain approximations of the eigenvalues of Γ, which are especially useful in case n is large; for small n an easy algorithm is available for exactly calculating the eigenvalues.

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In order to obtain information on the eigenvalues of Γ, we consider the tridiagonal $n\times n$ matrix

$$(1-\rho^2)\Gamma^{-1}=\begin{pmatrix} 1 & -\rho & 0 & \cdots & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & \ddots & & \vdots \\ 0 & -\rho & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1+\rho^2 & -\rho \\ 0 & \cdots & 0 & & -\rho & 1 \end{pmatrix}. \tag{1}$$

It seems reasonable to compare the eigenvalues μ_k of $(1-\rho^2)\Gamma^{-1}$ with those of a slightly perturbed matrix, chosen in such a way that its eigenvalues can be given explicitly. With this in mind, we consider the $n\times n$ matrices

$$M^{\text{sign}(\delta)}=M_{n,\rho}^{\text{sign}(\delta)}:=(1-\rho^2)\Gamma^{-1}+\rho(\rho+\delta)E, \tag{2}$$

where $\delta=-1$, 0 or 1, and $E:=\text{diag}(1,0,\ldots,0,1)$. The eigenvalues $\nu_k^{\text{sign}(\delta)}$ and corresponding eigenvectors of $M^{\text{sign}(\delta)}$ can be given explicitly, as we shall see in section 2. It is reasonable to expect that the Rayleigh Quotient of the normalized eigenvector x_k corresponding to $\nu_k^{\text{sign}(\delta)}$ with respect to the matrix $(1-\rho^2)\Gamma^{-1}$ [cf. Wilkinson (1965, ch. 3, §54)], i.e.,

$$\zeta_k^{\text{sign}(\delta)}:=x_k'(1-\rho^2)\Gamma^{-1}x_k,$$

provides a better approximation to μ_k than $\nu_k^{\text{sign}(\delta)}$ does. Hence we are led to the following approximations of the eigenvalues μ_k of $(1-\rho^2)\Gamma^{-1}$ (the μ_k's are arranged in increasing order as $\rho>0$ and in decreasing order as $\rho<0$):

$$\begin{aligned}
\zeta_k^- &=(1-\rho)^2+\frac{2}{n}\rho(1-\rho), && k=1,\\
&=1-2\rho\cos\frac{(k-1)\pi}{n}+\rho^2+\frac{2}{n}\rho(1-\rho)\left(1+\cos\frac{(k-1)\pi}{n}\right), && k=2,\ldots,n,\\
\zeta_k &=1-2\rho\cos\frac{k\pi}{n+1}+\rho^2-\frac{4}{n+1}\rho^2\left(\sin\frac{k\pi}{n+1}\right)^2, && k=1,\ldots,n, \quad (3)\\
\zeta_k^+ &=1-2\rho\cos\frac{k\pi}{n}+\rho^2-\frac{2}{n}\rho(1+\rho)\left(1-\cos\frac{k\pi}{n}\right), && k=1,\ldots,n-1,\\
&=(1+\rho)^2-\frac{2}{n}\rho(1+\rho), && k=n.
\end{aligned}$$

The corresponding error bounds for μ_k are given by

$$
\begin{aligned}
\varepsilon_k^- &= \left(\frac{2}{n}\right)^{\frac{1}{2}}(1-\rho), && k=1, \\
&= \left(\frac{4}{n}\right)^{\frac{1}{2}}\rho(1-\rho)\cos\frac{(k-1)\pi}{2n}, && k=2,\ldots,n, \\
\varepsilon_k &= \left(\frac{4}{n+1}\right)^{\frac{1}{2}}\rho^2\sin\frac{k\pi}{n+1}, && k=1,\ldots,n, \qquad (4) \\
\varepsilon_k^+ &= \left(\frac{4}{n}\right)^{\frac{1}{2}}\rho(1+\rho)\sin\frac{k\pi}{2n}, && k=1,\ldots,n-1, \\
&= \left(\frac{2}{n}\right)^{\frac{1}{2}}\rho(1+\rho), && k=n.
\end{aligned}
$$

Because $\lambda_k=(1-\rho^2)/\mu_k$, the error bounds for λ_k are accordingly given by expressions like $(1-\rho^2)\varepsilon_k/\zeta_k^2$, where if necessary + or − superscripts are added.

These error bounds show that for increasing n the approximations improve. For small values of n, it is not difficult to find good estimates by direct calculation, as we shall see in the next section.

2. Exact evaluation of eigenvalues

In this section we present an algorithm for the exact computation of the eigenvalues μ_k of $(1-\rho^2)\Gamma^{-1}$ and thus also of the eigenvalues λ_k of Γ, because $\lambda_k=(1-\rho^2)/\mu_k$. The algorithm is based on well-known facts, most of which are easily accessible in the literature [cf. Grenander and Szegö (1958, ch. 5)]. Nevertheless we shall provide short proofs because they have a bearing on the arguments used in the next section.

Assume that μ is an eigenvalue of $(1-\rho^2)\Gamma^{-1}$. We introduce α by putting $\mu=1+2\rho\alpha+\rho^2$. Then

$$
0=\det(\mu I-(1-\rho^2)\Gamma^{-1})
$$

$$
=\begin{vmatrix}
2\rho\alpha+\rho^2 & \rho & 0 & \cdots & \cdots & 0 \\
\rho & 2\rho\alpha & \rho & \ddots & & \vdots \\
0 & \rho & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & 2\rho\alpha & \rho \\
0 & \cdots & \cdots & 0 & \rho & 2\rho\alpha+\rho^2
\end{vmatrix} =: \rho^n \Delta_n^*(\alpha).
$$

Consequently, the values of μ are in one-to-one correspondence with the roots of the equation $\Delta_n(x)=0$ in the variable x.

In order to obtain a manageable expression for $\Delta_n(x)$, we proceed as follows. For each $n\in N$, let $U_n(x)$ be the unique nth degree Chebyshev polynomial of the second kind, normalized by $U_n(1)=n+1$. It is a well-known fact [for general information on the Chebyshev polynomials, see Abramowitz and Stegun (1965) and Rivlin (1974)] that $U_n(x)$ may be written as

$$U_n(x)=\begin{vmatrix} 2x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2x \end{vmatrix}.$$

By direct evaluation of the defining determinant for $\Delta_n(x)$, we find that

$$\Delta_n(x)=U_{n-2}(x)\rho^2+2U_{n-1}(x)\rho+U_n(x),$$

identically in ρ and x. In order to locate the roots of $\Delta_n(x)=0$, we consider the zero's of the Chebyshev polynomials $U_n(x)$. We define

$$\xi_k^{(n+1)}:=\cos\frac{k\pi}{n+1} \quad \text{for} \quad k=0,1,\ldots,n+1 \quad \text{and} \quad n\in N.$$

Since $U_n(x)=\sin(n+1)\theta/\sin\theta$ if $x=\cos\theta\,(-1<x<1)$, it follows that

$$U_n(\xi_k^{(n+1)})=0 \quad \text{for} \quad k=1,\ldots,n.$$

Now we are in a position to prove:

Proposition 1. The polynomial Δ_n of degree n has exactly n simple zero's $\alpha_1,\ldots,\alpha_n$ in the interval $(-1,1)$. If $-1<\alpha_1<\alpha_2<\ldots<\alpha_n<1$, then for $k=1,\ldots,n$ we have

$$\alpha_k\in I_k^+:=(\xi_{n-k+1}^{(n)},\xi_{n-k+1}^{(n+1)}) \quad \text{if} \quad 0<\rho<1,$$

$$\alpha_k\in I_k^-:=(\xi_{n-k+1}^{(n+1)},\xi_{n-k}^{(n)}) \quad \text{if} \quad -1<\rho<0.$$

Moreover,

$$\lim_{\rho\downarrow -1}\alpha_k=\xi_{n-k}^{(n)}, \qquad \lim_{\rho\to 0}\alpha_k=\xi_{n-k+1}^{(n+1)}, \qquad \lim_{\rho\uparrow 1}\alpha_k=\xi_{n-k+1}^{(n)}.$$

Proof. The Chebyshev polynomials $U_n(x)$ satisfy the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

Since

$$\Delta_n(x) = U_{n-2}(x)\rho^2 + 2U_{n-1}(x)\rho + U_n(x), \quad U_{n-1}(\xi_k^{(n)}) = 0 \quad \text{for} \quad k = 1, \ldots, n-1,$$

we find that for these values of k,

$$\Delta_n(\xi_k^{(n)}) = (-1)^k (1 - \rho^2),$$

$$\Delta_n(1) = n(1+\rho)^2 + 1 - \rho^2,$$

$$\Delta_n(-1) = (-1)^n \{n(1-\rho)^2 + 1 - \rho^2\}.$$

Consequently,

$$\Delta_n(\xi_{k-1}^{(n)}) \cdot \Delta_n(\xi_k^{(n)}) < 0 \quad \text{for} \quad k = 1, \ldots, n.$$

This shows that Δ_n changes sign at least n times on the interval $(-1, 1)$. Because Δ_n is a polynomial of degree n, this means that Δ_n has exactly n simple zero's in $(-1, 1)$. Moreover,

$$\alpha_{n-k+1} \in J_k := (\xi_k^{(n)}, \xi_{k-1}^{(n)}) \quad \text{for} \quad k = 1, \ldots, n.$$

Because $(k-1)/n < k/(n+1) < k/n$ for $k = 1, \ldots, n$, we see that

$$\xi_k^{(n+1)} \in J_k.$$

Furthermore,

$$\Delta_n(\xi_k^{(n+1)}) = 2\rho(-1)^{k-1}(1 + \rho\xi_k^{(n+1)}).$$

Hence

$$\Delta_n(\xi_k^{(n+1)}) \cdot \Delta_n(\xi_k^{(n)}) = -2\rho(1-\rho^2)(1 + \rho\xi_k^{(n+1)})$$

and ρ have opposite signs. This proves that

$$\alpha_k \in I_k^+, \quad 0 < \rho < 1 \quad \text{and} \quad \alpha_k \in I_k^-, \quad -1 < \rho < 0.$$

Clearly, α_k is a continuous function of ρ. Using the defining recurrence relation for the Chebyshev polynomials, the limits $\lim_{\rho \to \delta} \alpha_k$ are now easily calculated. Q.E.D.

These results make it possible to construct a simple algorithm for the direct computation of the μ's. Indeed, this proposition gives a set of n disjoint intervals I_k^+ (or I_k^-) with the following properties:

(i) each interval contains precisely one (simple) zero of Δ_n, and
(ii) the signs of the values of Δ_n at the end points of each interval are opposite.

Thus the rule of False Position (Regula Falsi) is suitable for application in each interval. In order to avoid slow convergence, which may happen when an interval is reached on which the function Δ_n is convex or concave, a refinement of the Regula Falsi method can be used, for instance the so-called Illinois algorithm [see Rabinowitz (1970, p. 25)].

We programmed the TI-59 pocket calculator accordingly and we found that for given n, ρ and precision $\eta=10^{-10}$, it took approximately one minute computation time to calculate each μ, independent of the size of the matrix Γ.

3. Approximations and error bounds

In order to approximate the eigenvalues of the matrix $(1-\rho^2)\Gamma^{-1}$, we need the eigenvalues and corresponding eigenvectors of the matrices $M^{\text{sign}(\delta)}=(1-\rho^2)\Gamma^{-1}+\rho(\rho+\delta)E$ explicitly.

Let $v=1+2\rho\beta+\rho^2$ be an eigenvalue of $M^{\text{sign}(\delta)}$. As in the previous section, we deduce that

$$U_n(\beta)=0, \quad \delta=0 \quad \text{and} \quad (\beta+\delta)U_{n-1}(\beta)=0, \quad \delta=\pm 1.$$

This yields:

Proposition 2. The matrix $M^{\text{sign}(\delta)}$ has eigenvalues $v_k^{\text{sign}(\delta)}$, $k=1,\dots,n$, with corresponding orthogonal eigenvectors $x_k^{\text{sign}(\delta)}$ with components $x_{jk}^{\text{sign}(\delta)}$, $j=1,\dots,n$:

$\underline{\delta=-1}$

$$v_k^- = 1+2\rho\xi_{n-k+1}^{(n)}+\rho^2 = 1-2\rho\cos\frac{(k-1)\pi}{n}+\rho^2,$$

$$x_{jk}^- = \cos\frac{(2j-1)(k-1)\pi}{2n},$$

$$\|x_1^-\|_2 = n^{\frac{1}{2}}, \quad \|x_k^-\|_2 = (\tfrac{1}{2}n)^{\frac{1}{2}} \quad \text{for} \quad k\neq 1.$$

$\underline{\delta=0}$

$$\nu_k = 1 + 2\rho\xi_{n-k+1}^{(n+1)} + \rho^2 = 1 - 2\rho\cos\frac{k\pi}{n+1} + \rho^2,$$

$$x_{jk} = \sin\frac{kj\pi}{n+1},$$

$$\|x_k\|_2 = (\tfrac{1}{2}(n+1))^{\frac{1}{2}}.$$

$\underline{\delta=1}$

$$\nu_k^+ = 1 + 2\rho\xi_{n-k}^{(n)} + \rho^2 = 1 - 2\rho\cos\frac{k\pi}{n} + \rho^2,$$

$$x_{jk}^+ = \sin\frac{(2j-1)k\pi}{2n},$$

$$\|x_k^+\|_2 = (\tfrac{1}{2}n)^{\frac{1}{2}} \quad \text{for} \quad k \neq n, \quad \|x_n^+\|_2 = n^{\frac{1}{2}}.$$

We omit the derivation of the eigenvectors. Although somewhat tedious, the calculations involved are elementary. Note that M $(\delta=0)$ is the $n \times n$ covariance matrix of a moving average process of order one and that ν_k given in Proposition 2 agrees with the value given in Grenander and Szegö (1958, p. 67).

It is not difficult to see that the cases $\delta=1$ and $\delta=-1$ are very similar. Indeed, changing ρ into $-\rho$ and k into $n-k+1$ has the effect of transforming $\delta=1$ into $\delta=-1$ and vice versa.

From now on we shall, for obvious reasons of simplicity, drop the sign of δ whenever there is no risk of confusion. So ν_k can also mean ν_k^+ or ν_k^-, etc.

Using ν_k as a first approximation of μ_k, standard perturbation theory indicates that the Rayleigh Quotient of x_k with respect to the matrix $(1-\rho^2)\Gamma^{-1}$, i.e.,

$$\zeta_k = \zeta(x_k) = x_k'(1-\rho^2)\Gamma^{-1}x_k/\|x_k\|_2^2,$$

is likely to give a better approximation to the eigenvalue μ_k of $(1-\rho^2)\Gamma^{-1}$ than ν_k (the corresponding indices of ζ_k, μ_k and ν_k indicate that their orderings agree); see Wilkinson (1965, ch. 3, §54, p. 172). Clearly,

$$\zeta_k = \nu_k - \rho(\rho+\delta)(x_{1k}^2 + x_{nk}^2)/\|x_k\|_2^2,$$

and this expression directly leads to (3).

To establish the error bounds in (4) we proceed as follows.

From now on we assume ρ to be positive. This gives no loss of generality, because changing ρ into $-\rho$ can be effected by changing the sign of δ and reversing the order of magnitude of the eigenvalues, e.g., $v_k^+(-\rho)=v_{n-k+1}^-(\rho)$, $\mu_k(-\rho)=\mu_{n-k+1}(\rho)$ and also $\zeta_k^+(-\rho)=\zeta_{n-k+1}^-(\rho)$. Since we have assumed ρ to be positive $(0<\rho<1)$, the eigenvalues v_k of M as given in Proposition 2 are arranged in increasing order. If we agree to a similar ordering of the μ's $(\mu_1<\mu_2<\ldots<\mu_n)$, then by the Courant–Fischer theorem [cf. Ortega (1972, 3.3.2, p. 57)] we have

$$|v_k-\mu_k|\leqq\|\rho(\rho+\delta)E\|_2=\rho(\rho+\delta)\quad\text{for}\quad k=1,\ldots,n.$$

Moreover, it is an easy consequence of the Wielandt–Hoffman theorem [cf. Wilkinson (1965, ch. 2 §48, p. 104)] that

$$\min_k|v_k-\mu_k|\leqq\left(\frac{2}{n}\right)^{\frac{1}{2}}\rho(\rho+\delta).$$

This gives approximately the right upper bound for all absolute differences $|v_k-\mu_k|$ as we shall see presently. Define

$$\varepsilon_k:=\|\{(1-\rho^2)\Gamma^{-1}-v_kI_n\}x_k/\|x_k\|_2\|_2.$$

This way we obtain formulae (4).

From elementary linear algebra we use the following result:

Let A be a symmetric $n\times n$ matrix and let $x\in\boldsymbol{R}^n$ be a given vector normalized by $\|x\|_2=1$. Then for every $\lambda_0\in\boldsymbol{R}$,

$$\min_\lambda|\lambda-\lambda_0|\leqq\|Ax-\lambda_0x\|_2,$$

where λ runs through all eigenvalues of A. Moreover, $\|Ax-\lambda x\|_2$ is minimized by $\lambda=x'Ax$.

A proof may be found in, e.g., Fox (1964, p. 279).

Application of this result yields

$$\min_i|\mu_i-v_k|\leqq\varepsilon_k,$$

and thus

$$|\mu_k-v_k|\leqq\varepsilon_k\quad\text{and}\quad|\mu_k-\zeta_k|\leqq\varepsilon_k,$$

provided ε_k is small enough.

Clearly, to proceed from here, we need additional information on the separation of the eigenvalues μ_k of $(1-\rho^2)\Gamma^{-1}$. Or, as we do not know the μ's, we could try the computable quantities defined by

$$\eta_k := \min_{i \neq k} |\zeta_i - \zeta_k|.$$

Indeed, $\eta_k > 2\varepsilon_k$ for all k, implies that the ε_k-neighbourhoods of the ζ_k's are mutually disjoint and hence

$$|\mu_k - \zeta_k| \leqq \varepsilon_k \quad \text{for all } k. \tag{5}$$

To improve upon this estimate, we assume that $\eta_k > 2\varepsilon_k$ for all k. Consequently, if $\varepsilon \geqq \max_k \varepsilon_k$ and $\min_k \eta_k > 2\varepsilon$, we have

$$\min_{i \neq k} |\mu_i - \zeta_k| \geqq \eta_k - \varepsilon > \varepsilon.$$

Now let $\{y_1, \ldots, y_n\}$ be an orthonormal basis of $\boldsymbol{R}^n$ consisting of eigenvectors of $(1-\rho^2)\Gamma^{-1}$, so that

$$(1-\rho^2)\Gamma^{-1} y_k = \mu_k y_k.$$

Put

$$x_k / \|x_k\|_2 = \sum_{i=1}^{n} \alpha_{ik} y_i \quad \text{for all } k.$$

Then

$$\sum_{i=1}^{n} \alpha_{ik}^2 = 1 \quad \text{and} \quad \zeta_k = \sum_{i=1}^{n} \mu_i \alpha_{ik}^2.$$

Now

$$\varepsilon_k^2 \geqq \|\{(1-\rho^2)\Gamma^{-1} - \zeta_k I_n\} x_k / \|x_k\|_2\|_2 = \sum_{i=1}^{n} (\mu_i - \zeta_k)^2 \alpha_{ik}^2$$

$$\geqq (\eta_k - \varepsilon) \cdot \sum_{i=1, i \neq k}^{n} |\mu_i - \zeta_k| \alpha_{ik}^2 \geqq (\eta_k - \varepsilon) |\mu_k - \zeta_k| \alpha_{kk}^2.$$

Hence, for $k = 1, \ldots, n$, we have $|\mu_k - \zeta_k| \leqq \varepsilon_k^2 / (\eta_k - \varepsilon) \alpha_{kk}^2$ and this gives

$$|\mu_k - \zeta_k| \leqq \varepsilon_k^2 / (\eta_k - 2\varepsilon), \tag{6}$$

because

$$\alpha_{kk}^2 = 1 - \sum_{i=1, i\neq k}^{n} \alpha_{ik}^2 \geqq 1 - (\eta_k - \varepsilon)^{-2} \sum_{i=1}^{n} (\mu_i - \zeta_k)^2 \alpha_{ik}^2$$

$$\geqq 1 - (\eta_k - \varepsilon)^{-2} \varepsilon_k^2 \geqq 1 - (\eta_k - \varepsilon)^{-2} \varepsilon^2 > (\eta_k - 2\varepsilon)/(\eta_k - \varepsilon).$$

Obviously, if $\eta_k \gg \varepsilon$, then (6) gives a better upper bound than (5). From the expression for ζ_k it can be seen that basically η_k/ρ is independent of ρ. This enables us to express the error bound in terms of ρ. We summarize our results in the following theorem:

Theorem. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\Gamma_{n,\rho}$. Put $\mu_k = (1-\rho^2)/\lambda_k$, $k = 1, \ldots, n$. Let ζ_k and ε_k be corresponding quantities as in (3) and (4).
Define

$$\eta_k = \min_{i\neq k} |\zeta_i - \zeta_k| \quad \text{and} \quad \varepsilon = \max_k |\varepsilon_k|.$$

Assume

$$\eta := \min_k \eta_k > 2\varepsilon,$$

then

$$|\mu_k - \zeta_k| \leqq \min(\varepsilon_k, \varepsilon_k^2/(\eta_k - 2\varepsilon)), \qquad k = 1, \ldots, n.$$

This implies in particular, provided $\eta \gg \varepsilon$, that

(i) $\mu_k = \zeta_k^- + \mathcal{O}(\rho(1-\rho)^2), \qquad \rho \uparrow 1,$

(ii) $\mu_k = \zeta_k + \mathcal{O}(\rho^3), \qquad \rho \to 0,$

(iii) $\mu_k = \zeta_k^+ + \mathcal{O}(\rho(1+\rho)^2), \qquad \rho \downarrow -1,$

where the constants involved can be effectively computed.

On the whole, one may expect the Rayleigh Quotients $\zeta_k^{sign(\delta)}$ to give good approximations of μ_k, especially when n is not too small. For small values of the parameter ρ, the Rayleigh Quotient ζ_k is rather precise; if ρ is close to 1, it is advisable to use ζ_k^- and likewise one should take ζ_k^+ in case ρ is close to -1.

It is not difficult, using the Propositions 1 and 2 and the Theorem, to work out exact error bounds for the approximations $(1-\rho^2)/\zeta_k$ of λ_k. We leave the details to the interested reader.

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