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An Inequality for Yff's Analogue of the Brocard Angle of a Plane Triangle

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Let a, b and c be the lengths of the sides of a plane triangle (T). Then the unique positive real root u of the cubic equation

$$u^3 = (a-u)(b-u)(c-u),$$

called the analogue of the Brocard angle of (T), satisfies

$$1/u < 1/a + 1/b + 1/c < k/u \ ,$$

where the constant $k \approx 1.5085$ is the real zero of the polynomial $8x^3 - 40x^2 + 125x - 125$. The inequalities are best possible in the sense that the difference of both sides of each inequality can be made arbitrarily small by choosing a suitable triangle (T). The method of proof employed is non-geometric, it is based on the Kuhn-Tucker theory of nonlinear programming. Some graphical heuristics are also provided.

1. Introduction

The positive (or first) Brocard point of a plane triangle $(T) = ABC$ is the unique point Ω interior to (T) with the property that $<\Omega AB = <\Omega BC = <\Omega CA =: \omega$. Similarly, the negative (or second) Brocard point Ω' of (T) satisfies $<\Omega' BA = <\Omega' CB = <\Omega' AC =: \omega'$ and $\omega' = \omega$. The common value ω of these angles is known as the Brocard angle of (T). See figure 1 below.

The Brocard points have many interesting properties. A great deal of information is given in EMMERICH's monograph [1]; see also [2] and [5].

A pair of analogue points U and U' were introduced by PETER YFF in [6]. Let U be situated such that AU intersects the opposite side BC of (T) in A_1, BU intersects CA in B_1, CU intersects AB in C_1 and $AC_1 = BA_1 = CB_1 =: u$. The point U' is similarly defined by $AB_1' = BC_1' = CA_1' =: u'$. See figure 2 above. In fact U' is the isotomic conjugate of U. If such a point U exists then U' exists also and Ceva's theorem implies that $u' = u$ and

$$u^3 = (a-u)(b-u)(c-u) \ , \tag{1}$$

where a,b and c are the lengths of the sides of (T) opposite A,B and C

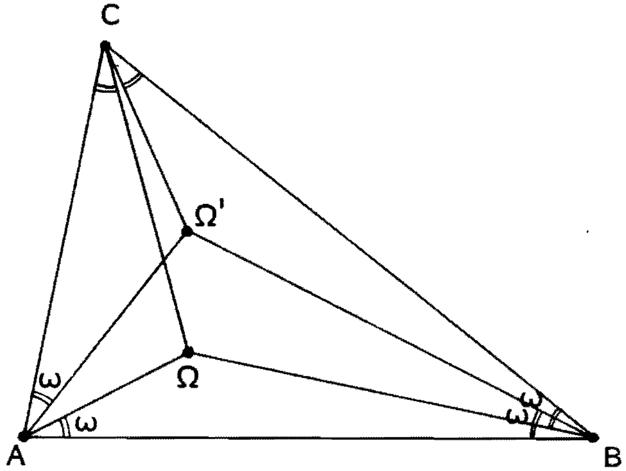

figure 1
The Brocard points Ω and Ω
and the Brocard angle ω

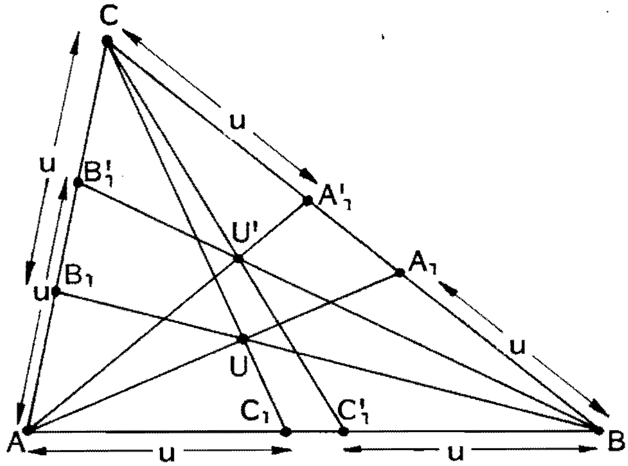

figure 2
The analogue Brocard points U and U'
and the analogue Brocard angle u

respectively. Conversely, any triple (a,b,c) of positive real numbers may serve to define a triangle (T), provided that $a+b>c, b+c>a$ and $c+a>b$. For any such triple (a,b,c) the cubic equation (1) has a uniquer real root u. Moreover, u is positive so that the point U is interior to the triangle (T) defined by a,b and c.

Summarizing, for any three positive numbers a,b and c with $a+b>c, b+c>a$ and $c+a>b$ a unique positive real number u exists, determined by (1), which shall be called the analogue of the Brocard angle of (T)

with sides of length a,b and c, for short $u = anbroc(a,b,c)$. All of this and much more can be found in YFF's paper [6].

Many relations between a,b,c and u can be established that have their analogues in Brocard geometry. A striking example is

$$8u^3 \leqslant abc\ . \tag{2}$$

This inequality inspired Yff to conjecture the remarkable inequality

$$8\omega^3 \leqslant \alpha\beta\gamma \tag{3}$$

(see [6], p.500). Here α,β and γ are the angles of a triangle with Brocard angle ω. For a proof of Yff's inequality the reader is referred to [5]. Inequality (2) is an immediate consequence of (1). Indeed,

$$64u^6 = 4u(a-u)\cdot 4u(b-u)\cdot 4u(c-u) \leqslant a^2b^2c^2\ ,$$

from which the required inequality follows directly. Of course exceptions to the analogy can be expected, because there is a difference in the way the sides of a triangle are restricted, whereas the angles are not.

In [5], searching for a sharpening of Yff's inequality (3), we were led to conjecture the following inequality

$$1/\omega < 1/\alpha + 1/\beta + 1/\gamma \leqslant 1.5/\omega\ . \tag{4}$$

Although we obtained massive numerical evidence, we could only prove the strict inequality in general and the remaining part of (4) in case of an isosceles triangle. Having failed to prove the Brocard inequality, we turned to the analogue and discovered that (4) almost remains intact in its analogue form, but not quite. The complete result is formulated in the following theorem.

THEOREM. *If a,b and c denote the lengths of the sides of a nondegenerate plane triangle (T) with $u = anbroc(a,b,c)$, then*

$$1/u < 1/a + 1/b + 1/c < k/u\ , \tag{5}$$

with $k = 1.5085$ approximately. Both inequalities are best possible in the sense that the difference of both sides of each inequality can be made arbitrarily small for a suitable choice of (T). The constant k is the unique real zero of the cubic polynomial $8x^3 - 40x^2 + 125x - 125$.

There is another inequality closely related to (4) which, unlike (4), carries over exactly to its true analogue form. This inequality may be found in [5] and its analogue is

$$\cdot 75/u^2 \leqslant 1/a^2 + 1/b^2 + 1/c^2 < 1/u^2\ , \tag{6}$$

with equality iff $a = b = c$.

In the sequel a detailed proof of the theorem shall be given using optimization techniques. Inequality (6) may be proved likewise. In [4] similar techniques were used to derive certain geometrical inequalities. It is unfortunate (perhaps) that the chosen method of proof is purely analytical, no geometrical arguments are used whatsoever. So some explanations and interpretations of a graphical nature shall be provided in the final section of this paper.

2. Analytical reformulation

Let $(T)=(a,b,c)$ be a plane triangle with sides of length a,b and c. Then $a+b>c, b+c>a, c+a>b$ and any triple of positive real numbers (a,b,c) satisfying these inequalities determines a triangle (T). Ordering the sides of (T) according to their magnitudes, the following necessary and sufficient conditions are obtained

$$a \geqslant b \geqslant c > 0 \text{ and } b+c>a . \tag{7}$$

For any triple (a,b,c) satisfying (7), define u to be the unique real root of the cubic equation (1). This root is necessarily positive, so that $u=anbroc(a,b,c)$. See figure 2. Put $s:=u/a+u/b+u/c$. It is not difficult to show (see [6]) that $2F(s-1)=ud$, where F denotes the area of (T) and d is the sum of the distances of U from the sides of (T). This shows that $s>1$. Alternatively, this follows from

$$abc(s-1) = u^2(a+b+c-2u) \text{ and } u<\min(a,b,c)\leqslant(a+b+c)/3 .$$

Clearly, u vanishes as c tends to 0 while a and b are kept constant. From (1) it then follows that c/u tends to 1 and consequently s also tends to 1. This shows 1 to be the best possible lower bound for s. The easiest part of the theorem has thus been proven. Next we turn to investigate the best possible upper bound for s.

Allowing for degenerate triangles $(T)=(a,b,c)$ with $a\geqslant b\geqslant c\geqslant 0$ and $b+c=a$, including the case $c=0, b=a\neq 0$ but excluding $a=b=c=0$, the conditions (7) are extended to

$$a\geqslant b\geqslant c\geqslant 0,\ b+c\geqslant a \text{ and } (a,b,c)\neq(0,0,0) . \tag{8}$$

Put $x:=u/a, y:=u/b, z:=u/c$ such that $z=1$ when $c=0$. Then

$$0\leqslant x\leqslant y\leqslant z\leqslant 1,\ x(y+z)\geqslant yz,\ xyz = (1-x)(1-y)(1-z)$$

by (1) and (8). Moreover, $s=x+y+z$.

Thus the theorem - or rather the non-trivial, so far unproven part of it - can be rephrased as the following optimization problem:

MAIN PROBLEM

maximize $s=x+y+z$

subject to

$$C: \begin{cases} x\geqslant 0, y-x\geqslant 0, z-y\geqslant 0, 1-z\geqslant 0 \\ x(y+z)-yz\geqslant 0 \\ xyz-(1-x)(1-y)(1-z)=0 \end{cases}$$

Because of the polynomial form of both the objective function and the constraint set C, there is a good chance of determining an explicit solution by applying non-linear programming techniques to the main problem.

3. Kuhn-Tucker theory

Let $f, g_1, \cdots, g_n$ be real-valued functions defined on a subset X of $\mathbb{R}^n$. Optimization problems that can be put into the form

$$\text{maximize } f(x)$$
$$\text{subject to}$$
$$C: \begin{cases} g_i(x) \geqslant 0 \text{ for } i=1, \cdots, m \\ g_i(x)=0 \text{ for } i=m+1, \cdots, n \end{cases} \text{ and } x \in X \tag{9}$$

are the subject matter of what is known as programming, linear programming when all of the functions f, g_i are linear and non-linear programming otherwise. The constraint set C is supposed to be non-empty. If C is compact and f is continuous, as is obviously the case for the main problem with $f=s$, a well-known theorem of Weierstrass guarantees the existence of a solution to the programming problem (9). The question remains as to how this solution can be found. Sometimes the following theorem, named after its discoverers H.W. Kuhn and A.W. Tucker, turns out to be useful in this respect.

Kuhn-Tucker theorem

Consider problem (9) with totally differentiable functions f, g_i defined on an open subset X of $\mathbb{R}^n$. Further, for all elements x of C, let $E(x)$ be the set of indices $j \in \{1, \cdots, m\}$ for which $g_j(x)=0$. Now suppose that f attains a local maximum on C at x^*. Assume also that the system of gradient vectors $\{\nabla g_i(x^*): i \in E(x^*) \cup \{m+1, \cdots, n\}\}$ is linearly independent (this is a regularity condition called the *LI*-condition). Then real numbers $\lambda_1, \cdots, \lambda_n$ exist such that

$$\begin{aligned} &(i) \quad \nabla f(x^*) + \sum_{i=1}^{n} \lambda_i \nabla g_i(x^*) = 0 \text{ and} \\ &(ii) \quad \lambda_i g_i(x^*) = 0 \text{ for } i = 1, \cdots, n, \text{ where} \\ &\qquad g_i(x^*) \geqslant 0 \text{ and } \lambda_i \geqslant 0 \text{ for } i = 1, \cdots, m \text{ and} \\ &\qquad g_i(x^*) = 0 \text{ for } i = m+1, \cdots, n . \end{aligned} \tag{10}$$

Proofs of this theorem may be found in various places e.g. [3], p.314. There exists a wide variety of regularity conditions of which we have chosen the *LI*-condition as it is the obvious one for our purpose. The relations (10) are usually called the first order conditions or the Kuhn-Tucker conditions.

4. Proof of the theorem

Suppose that $x^*=(x,y,z)$ satisfies the first order conditions of the main problem. Then according to the Kuhn-Tucker theorem real numbers $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5$ and μ exist such that

$$\begin{aligned}
&1 + \lambda_1 - \lambda_2 + \lambda_5(y+z) + \mu[yz + (1-y)(1-z)] = 0,\\
&1 + \lambda_2 - \lambda_3 + \lambda_5(x-z) + \mu[xz + (1-x)(1-z)] = 0,\\
&1 + \lambda_3 - \lambda_4 + \lambda_5(x-y) + \mu[xy + (1-x)(1-y)] = 0,\\
&x \geqslant 0, y-x \geqslant 0, z-y \geqslant 0, 1-z \geqslant 0, x(y+z)-yz \geqslant 0, \qquad (11)\\
&\lambda_1 x = \lambda_2(y-x) = \lambda_3(z-y) = \lambda_4(1-z) = \lambda_5[x(y+z)-yz] = 0,\\
&\lambda_i \geqslant 0 \text{ for } i = 1,\cdots,5,\\
&xyz-(1-x)(1-y)(1-z) = 0.
\end{aligned}$$

If $x=0$ then $yz \leqslant 0$ and $(1-y)(1-z)=0$, which implies $y=0$ and $z=1$, because $z-y \geqslant 0$. Hence $z-y>0$ and thus $\lambda_3=0$. Adding the leading two equations of (11) shows that $\lambda_1=-2$, which is clearly impossible. Hence $x>0$. Then also $z<1$, since $z=1$ implies $xy=0$. Consequently $\lambda_1=\lambda_4=0$. Let us consider three cases separately, namely: $x=y, x\neq y=z$ and $x\neq y\neq z$.

$I: x = y$

If $x=y$ then $x(y+z)-yz=y^2>0$ and hence $\lambda_5=0$. The first three equations of (11) reduce to

$$\begin{aligned}
&1 - \lambda_2 + \mu[yz + (1-y)(1-z)] = 0,\\
&1 + \lambda_2 - \lambda_3 + \mu[yz + (1-y)(1-z)] = 0, \qquad (12)\\
&1 + \lambda_3 + \mu[y^2 + (1-y)^2] = 0.
\end{aligned}$$

The third equation of (12) implies $\mu\neq 0$ and subtracting the second equation from the first and the third equation gives

$$-2\lambda_2 + \lambda_3 = 0 \text{ and } -\lambda_2 + 2\lambda_3 + \mu(y-z)(2y-1) = 0\,. \qquad (13)$$

Since $y-z=0$ iff $2y-1=0$, which follows from the last equation of (11), $y\neq z$ cannot be true. Indeed, $y\neq z$ implies $\lambda_3=0$ and thus $\lambda_2=0$ forcing $\mu(y-z)(2y-1)$ to vanish, by (13). This means $y=1/2=z$, a contradiction. Hence $y=z$ and consequently $x=y=z=1/2, \lambda_2=\lambda_3=0$ and $\mu=-2$. This completes case I. The only possible point is

$$x^* = (1/2,1/2,1/2),\ \lambda_i = 0 \text{ for } i = 1,2,3,4,5\,,$$

$$\mu = -2 \text{ and } s = 1.5.$$

$II: x\neq y = z$

Since $x\neq y=z$ it follows that $\lambda_2=0$ and $2xz-z^2\geqslant 0$ or $2x\geqslant z$, because $z\geqslant x>0$. Also $xz^2=(1-x)(1-z)^2$. The first equations of (11) now reduce to

$$\begin{aligned}
&1 + 2\lambda_5 z + \mu[z^2 + (1-z)^2] = 0\,,\\
&1 - \lambda_3 + \lambda_5(x-z) + \mu[xz + (1-x)(1-z)] = 0\,, \qquad (14)\\
&1 + \lambda_3 + \lambda_5(x-z) + \mu[xz + (1-x)(1-z)] = 0.
\end{aligned}$$

Clearly $\lambda_3=0$. Multiplying the first equation of (14) by $x(1-x)$ and the second one by $z(1-z)$, using $xz^2=(1-x)(1-z)^2$, leads to

$$x(1-x) + 2\lambda_5 xz(1-x) + \mu xz^2 = 0 \text{ and}$$

$$z(1-z) + \lambda_5 z(1-z)(x-z) + \mu xz^2 = 0\,.$$

If $\lambda_5=0$ then $x(1-x)=z(1-z)$ or $x+z=1$ as $x=z$. However, this would imply $(1-z)z^2=z(1-z)^2$ or $z=1/2$, which contradicts $x\neq z$. So $\lambda_5>0$ and hence $2x-z=0$ by (11). On substituting $x=z/2$ into $xz^2=(1-x)(1-z)^2$, the following cubic equation in the variable z is obtained

$$2z^3 - 4z^2 + 5z - 2 = 0\,. \tag{15}$$

This equation has a single real root $z=.6034$ approximately. Checking the values of λ_5 and μ results in $\lambda_5=.0877$ approximately and $\mu=-2.1210$ approximately.
The critical point found is

$$x^* = (z/2,z,z),\lambda_i = 0 \text{ for } i = 1,2,3,4\,,$$

$$\lambda_5 = (2-3z)/[2z(3-2z)]\,,$$

$$\mu = -5(1-z)(2-z)/[2z^2(3-2z)] \text{ and } s = 5z/2\,,$$

where z is the real root of (15) so that $\lambda_5>0$ and $s=1.5085$ approximately in this point.

III: $x\neq y\neq z$.

Clearly $\lambda_2=\lambda_3=0$ by (11). On multiplying the first, second and third equation of (11) by $1-x, 1-y$ and $1-z$ respectively, applying at the same time the final equation of (11), we get

$$1-x + \lambda_5(1-x)(y+z) + \mu yz = 0\,,$$

$$1-y + \lambda_5(1-y)(x-z) + \mu xz = 0\,,$$

$$1-z + \lambda_5(1-z)(x-y) + \mu xy = 0\,,$$

If we assume $\lambda_5=0$ then $-\mu xyz=x(1-x) = y(1-y) = z(1-z)$. Since $x\neq y$ and $y\neq z$ we deduce that $x+y=y+z=1$ and hence $x=z$. But then $x=y=z$ because of $x\leqslant y\leqslant z$ and this contradicts our supposition. Hence $\lambda_5>0$ so that $x(y+z)-yz=0$. Summarizing, this leaves the following system to be solved:

$$\begin{aligned}
&1-x + \lambda_5(1-x)(y+z) + \mu yz = 0,\\
&1-y + \lambda_5(1-y)(x-z) + \mu xz = 0,\\
&1-z + \lambda_5(1-z)(x-y) + \mu xy = 0,\\
&\lambda_5>0,\ 0<x<y<z<1,\\
&x(y+z) - yz = 0,\\
&xyz - (1-x)(1-y)(1-z) = 0.
\end{aligned} \tag{16}$$

Elimination of λ_5 and μ from (16) leads to a system of three equations in the three unknowns x,y and z. For reasons of simplicity, we subtract the second from the third equation of (16) and divide through by $y-z$ to obtain

$$1 + \lambda_5(x-1) + \mu x = 0\,. \tag{17}$$

Further, dividing the second and third equation of (16) through by $1-y$ and $1-z$ respectively and subtracting the resulting equations, leaves, after division by $y-z$,

$$\lambda_5 + \mu x(y+z-1)/[(1-y)(1-z)] = 0\,. \tag{18}$$

Now
$x(y+z-1)=x(y+z-1-yz+yz)=-x(1-y)(1-z)+xyz=-x(1-y)(1-z)+(1-x)(1-y)(1-z)=(1-2x)(1-y)(1-z)$, because of the last equation of (16). Hence (18) may be written as

$$\lambda_5 + \mu(1-2x) = 0\,. \tag{19}$$

Solving (17) and (19) for λ_5 and μ gives

$$\lambda_5 = (-2x+1)/(2x^2-2x+1),\ \mu = -1/(2x^2-2x+1)\,,$$

so that in particular

$$\mu + 1 = -\lambda_5 x\,. \tag{20}$$

The information contained in the first and one but last equations of (16) has not been used so far. If the first equation of (16) is divided though by $1-x$, we have

$$\begin{aligned} 0 &= 1 + \lambda_5(y+z) + \mu[yz+(1-y)(1-z)] \\ &= 1 + [\lambda_5+\mu(2x-1)](y+z) + \mu\,, \end{aligned}$$

by observing that $x(y+z)=yz$. Application of (19) yields

$$\mu + 1 = -2\lambda_5(y+z)\,.$$

Combining this with (20) gives

$$y + z = x/2\,.$$

But this is clearly impossible, since $y+z>2x>0$.

SUMMARY. Only two solutions $x^*=(x,y,z)$ of the Kuhn-Tucker conditions (11) of the main problem exist, namely
(i) $x_1^* = (1/2,1/2,1/2)$ with $\lambda_i = 0$ for $i=1,2,3,4,5$, $\mu=-2$ and $s=1.5$,
(ii) $x_2^* = (z/2,z,z)$ with $\lambda_i=0$ for $i=1,2,3,4$,

$$\lambda_5 = (2-3z)/[2z(3-2z)]\approx .0877, \mu = -5(1-z)(2-z)/[2z^2(3-2z)]$$

≈ -2.1210 and $s = 5z/2\approx 1.5085$, where z is the real root of the cubic equation (15).

At both points the LI-condition is satisfied. This is obvious for x_1^* as

$E(x_1^*)=\{2,3,6\}$ and $\nabla g_2=(-1,1,0), \nabla g_3=(0,-1,1), \nabla g_6=(1/2,1/2,1/2)$. Note that the subscript i of g_i corresponds to the i-th constraint as in the main problem. At x_2^* we have $E(x_2^*)=\{3,5,6\}$ and $\nabla g_3=(0,-1,1), \nabla g_5=(2z,-z/2,-z/2), \nabla g_6=(2z^2-2z+1, z^2-3z/2+1, z^2-3z/2+1)$. These gradients are also linearly independent, as is easily checked.

Consequently, if the (absolute) maximum of s is attained at $x^* \in C$, then either $x^*=x_2^*$ (because the s-value of x_2^* is larger than the s-value of x_1^*), or the LI-condition at x^* is violated. However, (0,0,1) is the only point of the constraint set C at which this is so and its s-value is too small to qualify, namely 1. This completes the proof of the theorem.

5. Graphical heuristics

Using yet another normalization on putting $x:=a/u-1, y:=b/u-1$, $z:=c/u-1$ and $p:=(a+b+c)/u-3$, equation (1) is transformed into $xyz=1$. Moreover, $x+y+z=p$ and the condition $a+b>c$ becomes $p+1>2z$.

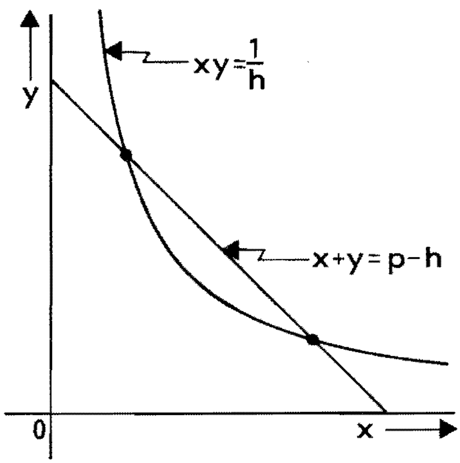

figure 3 ($p=3.4, h=1.2$)
Intersection of $xy=1/h$ and
$x+y=p-h$ in the plane $z=h$

A straightforward application of the inequality of the arithmic and geometric means gives

$$1 = (xyz)^{\frac{1}{3}} \leqslant (x+y+z)/3 = p/3 \text{ or } p \geqslant 3 .$$

In the new normalization we have

$$\begin{aligned} &x+y+z=p, \\ &xyz=1, \\ &0<2x, 2y, 2z<p+1, \\ &p\geqslant 3. \end{aligned} \tag{21}$$

Now let $p \geqslant 3$ be fixed. For any constant $h \in (0,p)$, the intersection of the hyperbola, given by the equations $xy = 1/h$ and $z = h$, with the plane $x + y + z = p$, consists of two (possibly coinciding) points or no intersection occurs (see figure 3). Intersection does occur if h satisfies the inequality

$$(p - h)^2 \geqslant 4/h$$

or

$$p \geqslant h + 2/\sqrt{h} =: \phi(h) .$$

The function $\phi:(0,\infty) \rightarrow \mathbb{R}$ behaves as shown in figure 4 below, so that intersection occurs on a full h-interval with interior point 1, except when $p = 3$ in which case intersection occurs only for $h = 1$.

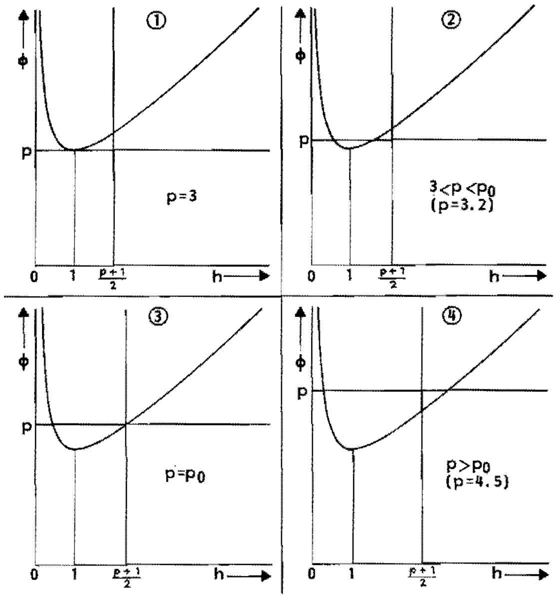

figure 4 ($p_0 \approx 3.6292$)
The function $\phi(h) = h + 2/\sqrt{h}$

Collecting all intersection points in the plane $x + y + z = p$ as the z-value h runs through the interval $(0,p)$, a closed curve is obtained. In figure 5 below, several such p-curves are shown. These p-curves may be interpreted as follows. Let $U(T)$ be the set of all triangles with unit perimeter. Note that the ratio of the perimeters of two congruent triangles equals that of their respective analogue Brocard angles. Now each point on a p-curve corresponds uniquely to a triangle of $U(T)$ with constant analogue Brocard angle $u = 1/(p + 3)$. For $p = 3$ only one such

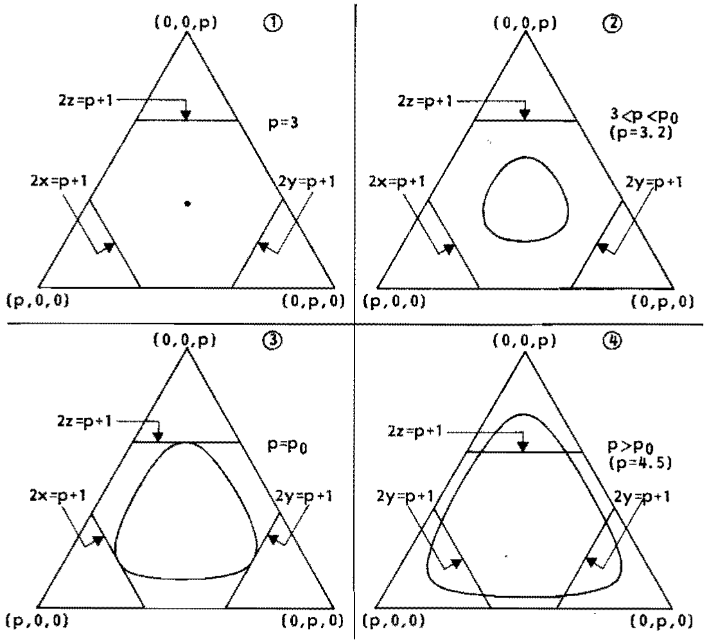

figure 5
($p_0 \approx 3.6292$)
p-curves: $x+y+z=p, xyz=1$

$U(T)$-triangle exists, namely the unilateral one, and for $p>3$ there are inifinitely many such triangles. Conversely, any two triangles of $U(T)$, having the same analogue Brocard angle, lie on the same p-curve.

One restriction has not been mentioned yet. That is: x,y and z should remain below the bound of $(p+1)/2$ (cf. (21)). As a consequence, certain points on a p-curve may be inadmissable. As mentioned before, the p-curve for $p=3$ consists of the single point $(x,y,z)=(1,1,1)$. This point is clearly admissable. Because of continuity, no points on the p-curve will violate the $(p+1)/2$ bound, provided p is close enough to the value 3. The critical p-value occurs when

$$p = \phi((p+1)/2) \text{ or } p^3-p^2-p-31 = 0 .$$

This value is $p_0=3.6292$ approximately. So, if $3<p<p_0$, picture 5.2 is relevant, and the case $p>p_0$ is illustrated by picture 5.4.

Let us return to the main problem, that is the inequality

$$u/a + u/b + u/c < k ,$$

rewritten as

$$1/(x+1) + 1/(y+1) + 1/(z+1) < k .$$

For $p>3$, we consider the Lagrange problem of the function

$$f_p(x,y,z) := 1/(x+1) + 1/(y+1) + 1/(z+1),$$

subject to the conditions

$x+y+z = p$ and $xyz = 1$,

on the open set $X := \{(x,y,z): x>0, y>0, z>0\} \subset \mathbb{R}^3$.

It is easy to check that the stationary points of the Lagrange problem are the six points given by

$$(i) \quad x = y = 1/\sqrt{z}, \phi(z) = p \text{ (2 points)},$$

$$(ii) \quad y = z = 1/\sqrt{x}, \phi(x) = p \text{ (2 points)},$$

$$(iii) \quad z = z = 1/\sqrt{y}, \phi(y) = p \text{ (2 points)}.$$

Because of symmetry, these points may be partitioned in two sets of three: at one set $M := \{M_x, M_y, M_z\}$ the maximum of f_p is attained and at the other set $m := \{m_x, m_y, m_z\}$, f_p attains its minimum. See figure 6.

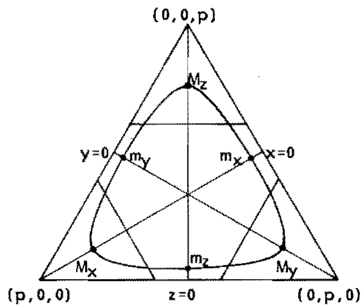

figure 6

Extremal points for f_p (22) on the p-curve: $x+y+z=p, xyz=1$

Now the value of f_p at points of M equals $M(p) := 2\sqrt{t}/(1+\sqrt{t}) + 1/(1+t)$, with $\phi(t)=p$ and $t>1$. Considered as a function of p, $M(p)$ increases with limiting value 2. However, the restriction $t<(p+1)/2$ prohibits $M(p)$ to come arbitrarily close to this value. As p runs through the interval $(3, p_0]$, $M(p)$ strictly increases to the ultimate value $M(p_0) = 10/(p_0+3) = k = 1.5085$ approximately, and the inequality $t<(p+1)/2$ remains true throughout this process. Beyond the value p_0 this inequality is violated. So, if $p>p_0$, the best (i.e. largest) value f_p attains at points of the p-curve is reached at the six points for which

$$(i) \quad x+y = (p-1)/2, z = (p+1)/2, xyz = 1 \text{ (2 points)},$$

$$(ii) \quad y+z = (p-1)/2, x = (p+1)/2, xyz = 1 \text{ (2 points)},$$

$$(iii) \quad z+x = (p-1)/2, y = (p+1)/2, xyz = 1 \text{ (2 points)}.$$

Again, see figure 6. At these points f_p attains the value

$$\tilde{M}(p) := q(q+1)/(q^2+1) + 1/(q+1) \text{ with } q = (p+1)/2 .$$

As a function of $p, \tilde{M}(p)$ strictly decreases on (p_0, ∞) and this suggests that $M(p_0)$ is indeed the largest value f_p attains for any $p>3$ if in addition to the conditions of (22) the arguments x, y and z also restricted by the inequalities

$$2x < p+1, \quad 2y < p+1, \quad 2z < p+1 .$$

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