Additions to:

On the Elliptic Logarithm Method for Elliptic Diophantine Equations: Reflections and an Improvement

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Abstract

This note contains some material not included in our paper [ST99]. This additional material comprises tables of all integer solutions to equations (15), (16), (17), and (19) of the paper cited.

1 Example 3: Top

The points P_i, for $i = 1, \ldots, 6$, in the table below are the elements of the c_1-optimal basis $\mathcal{B}_1$ in TABLE 2 of [ST99].

Table 1: All solutions to Top's quartic equation (15) of [ST99]

Quartic equation $y^2 = 24784x^4 + 90096x^3 + 114372x^2 + 1376352x + 7096896$ with integral points $P = [x, y] = \sum_{i=1}^{6} m_i P_i$, $y > 0$			
$[x, y]$	$(m_1, \ldots, m_6)$	$[x, y]$	$(m_1, \ldots, m_6)$
$[-493, 38122070]$	$(-1, 0, -1, 0, 1, 1)$	$[2, 3380]$	$(1, 0, 0, 0, 0, 1)$
$[-4, 2000]$	$(0, 1, -1, 1, 0, 0)$	$[3, 4086]$	$(0, 1, 0, -1, 0, 0)$
$[-3, 1890]$	$(1, 0, -1, 0, 0, 1)$	$[4, 5152]$	$(1, 1, -1, 1, 0, 0)$
$[-2, 2116]$	$(0, 0, -1, 0, 1, 1)$	$[12, 26640]$	$(0, 0, -1, 0, 0, 1)$
$[-1, 2402]$	$(1, 1, -1, -1, 0, 1)$	$[24, 97848]$	$(1, 2, -1, 0, 0, 0)$
$[0, 2664]$	$(0, 0, 0, 0, 0, 0)$	$[36, 214560]$	$(1, 0, -1, 0, 1, 1)$
$[1, 2950]$	$(0, 0, 0, 0, -1, 1)$	$[9636, 14620465440]$	$(0, 1, -2, 1, 0, 0)$

For the computations leading to all integer points on equation (16) of [ST99] as contained in the table below, we used basis $\mathcal{B}_1$ in TABLE 2 of [ST99], properly transformed to fit Weierstraß equation (16).

Table 2: All solutions to Top's equation (16) of [ST99]

Weierstraß equation $y^2 + xy + y = x^3 - x^2 - 28159452x + 15511281951$ with integral points $P = [x, y] = \sum_{i=1}^{6} m_i P_i$; of the additive inverses $[x, y]$ and $[x, -y - x - 1]$ only one is listed			
$[x, y]$	$(m_1, \ldots, m_6)$	$[x, y]$	$(m_1, \ldots, m_6)$
$[-5519, 55629]$	$(0, -1, 0, 0, -1, 0)$	$[20729, 2877289]$	$(1, 1, -1, -1, 0, 0)$
$[-5451, 86501]$	$(0, 1, 0, 0, 0, 1)$	$[26441, 4200569]$	$(0, -1, 0, 1, -1, -1)$
$[-5379, 109109]$	$(1, -1, 0, 0, 1, -1)$	$[27141, 4373189]$	$(1, 0, -1, 0, 0, 0)$
$[-5349, 116999]$	$(-1, 0, 0, -1, 0, 0)$	$[28671, 4758069]$	$(0, 1, 1, -1, 1, 1)$
$[-5099, 165349]$	$(-1, -1, 0, 0, 0, 0)$	$[31201, 5416749]$	$(0, 1, 0, 1, 0, 0)$
$[-4939, 187109]$	$(1, 0, -1, 1, -1, -1)$	$[32301, 5711549]$	$(0, -1, -1, -1, 0, -1)$
$[-4681, 213859]$	$(1, -1, 0, 0, 0, 0)$	$[38455, 7450597]$	$(-1, -1, 2, 1, 0, 0)$
$[-4499, 228349]$	$(0, 0, 1, 1, 0, 0)$	$[39701, 7820549]$	$(0, 0, 1, 0, 1, 1)$
$[-4219, 245429]$	$(0, 0, 0, -1, 1, 0)$	$[55071, 12836469]$	$(-1, -1, 0, 1, 0, -1)$
$[-3449, 269299]$	$(0, 1, -1, 0, -1, 0)$	$[60951, 14960549]$	$(1, 0, 0, 0, 0, -1)$
$[-3099, 271749]$	$(0, 0, 0, -1, 0, 1)$	$[80161, 22606129]$	$(0, 1, 0, 1, -1, 1)$
$[-2395, 264277]$	$(-1, 0, 1, 1, 0, 1)$	$[84821, 24612709]$	$(0, -1, -1, -1, -1, 0)$
$[-1895, 250077]$	$(0, 0, 0, 1, -1, -1)$	$[100501, 31766149]$	$(1, -1, 0, 1, 0, -1)$
$[-1579, 237509]$	$(0, -1, 1, 0, 0, 0)$	$[109741, 36256789]$	$(-1, 0, 0, 0, -1, 0)$
$[-1349, 226599]$	$(0, 1, 1, 0, 1, 1)$	$[115901, 39358349]$	$(0, 1, -1, -1, 0, 1)$

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Weierstraß equation $y^2 + xy + y = x^3 - x^2 - 28159452x + 15511281951$
with integral points $P = [x, y] = \sum_{i=1}^{6} m_i P_i$; of the additive inverses
$[x, y]$ and $[x, -y - x - 1]$ only one is listed

$[x, y]$	$(m_1, \ldots, m_6)$	$[x, y]$	$(m_1, \ldots, m_6)$
$[-1195, 218437]$	$(0, -1, -1, 0, 0, -1)$	$[133269, 48546021]$	$(0, 0, 1, 2, 0, -1)$
$[-759, 191289]$	$(-1, -1, 1, 0, 1, 0)$	$[141551, 53147899]$	$(1, 1, 0, 1, 1, 0)$
$[-691, 186461]$	$(0, 0, -1, -1, -1, 0)$	$[157205, 62216197]$	$(-1, 2, 0, 0, 0, 1)$
$[-589, 178879]$	$(1, 0, 0, 1, 0, -1)$	$[170249, 70127549]$	$(-1, 0, -1, -2, 0, 0)$
$[-249, 150149]$	$(-1, 0, 0, 1, -1, 0)$	$[178149, 75070101]$	$(-1, -1, 1, -1, 0, 1)$
$[181, 101989]$	$(1, 0, -1, -1, 0, 0)$	$[217541, 101324869]$	$(0, 0, 0, 0, 1, -1)$
$[261, 90309]$	$(0, 1, 0, 0, 0, 0)$	$[222301, 104671149]$	$(-1, -1, 0, 1, -1, 0)$
$[401, 65249]$	$(0, 0, 0, 1, 0, 0)$	$[274301, 143497549]$	$(1, 0, 0, 0, -1, 0)$
$[461, 51029]$	$(-1, -1, 0, 0, 0, -1)$	$[379131, 233232029]$	$(-1, 1, 0, 1, 0, 1)$
$[541, 20589]$	$(1, -1, 0, 0, 0, -1)$	$[447901, 299514349]$	$(0, -1, 1, -1, 2, 0)$
$[555, 7037]$	$(1, 1, 0, 0, 1, 0)$	$[501251, 354610349]$	$(1, 2, 0, 0, 0, 1)$
$[5021, 23669]$	$(1, 0, 0, 0, 0, 0)$	$[579301, 440608149]$	$(1, 0, -1, -2, 0, 0)$
$[5051, 43549]$	$(0, 1, 1, 1, 0, 0)$	$[644289, 516815441]$	$(0, -1, 0, 1, 1, -1)$
$[5301, 120549]$	$(0, 0, 0, 0, 1, 0)$	$[811701, 730875749]$	$(-2, 0, 0, 0, 0, 0)$
$[5701, 197749]$	$(0, 1, 0, -1, 0, 1)$	$[880861, 826269229]$	$(1, 0, -2, 0, -1, -1)$
$[5805, 215357]$	$(0, 1, -1, 1, -1, 0)$	$[934961, 903563009]$	$(2, 0, 0, 0, 1, -1)$
$[6121, 266109]$	$(0, 0, 0, 0, 0, 1)$	$[978141, 966887669]$	$(1, -1, 0, 1, -1, 0)$
$[6421, 312069]$	$(-1, -1, 0, -1, 0, 0)$	$[1239221, 1378871909]$	$(0, 1, 0, -1, 0, 0)$
$[6605, 339677]$	$(0, -1, 1, 1, 1, -1)$	$[2367381, 3641329029]$	$(0, 1, -1, 1, -1, -1)$
$[6759, 362579]$	$(-1, 1, 0, -1, 1, 1)$	$[2819371, 4732582269]$	$(0, 0, 1, 2, -1, 0)$
$[7251, 435149]$	$(1, -1, 0, -1, 0, 0)$	$[4049061, 8145601109]$	$(0, -2, 0, 0, -1, -1)$
$[7639, 492291]$	$(0, 0, 1, 0, 0, 0)$	$[4725451, 10269863799]$	$(-2, -1, 1, -1, 1, 1)$
$[8591, 634139]$	$(0, 0, -1, 0, 0, -1)$	$[5790801, 13932129549]$	$(1, -1, -1, -1, 0, 0)$
$[8661, 644709]$	$(0, -1, 1, 1, 0, 0)$	$[12785481, 45710379129]$	$(-1, 0, -1, -2, -1, 1)$
$[9509, 774661]$	$(0, -1, 0, -1, 1, 0)$	$[16209251, 65251492399]$	$(0, 0, 1, -2, 1, 1)$
$[10201, 883549]$	$(-1, 0, 1, 0, 1, 0)$	$[21178849, 97455492849]$	$(1, -2, 1, 2, 0, -2)$
$[10531, 936429]$	$(0, 1, -1, -1, -1, 0)$	$[30742931, 170442826829]$	$(-1, -1, 2, 1, 2, 0)$
$[12501, 1265349]$	$(0, 0, -1, 0, -1, 0)$	$[102867669, 1043270519909]$	$(0, 2, 1, 2, 0, 1)$
$[12581, 1279189]$	$(1, 0, 1, 0, 1, 0)$	$[150904501, 1853683618149]$	$(-1, -1, 1, 1, -1, -1)$
$[15801, 1867049]$	$(-1, 0, 1, 0, 0, 1)$	$[338266805, 6221244429157]$	$(0, 1, -2, -1, 0, -1)$
$[17691, 2239239]$	$(1, 1, 0, 1, -1, 0)$	$[1021640251, 32654286495399]$	$(-1, 0, -2, 0, -2, 0)$
$[19561, 2626329]$	$(-1, 0, -1, 0, 0, 0)$		

2 Example 4: Buddenhagen

In the table below, the points P_i, for $i = 1, \ldots, 7$, are the elements of the c_1-optimal basis $\mathcal{B}_2$ in TABLE 3 of [ST99].

Table 3: All solutions to Buddenhagen's equation (17) of [ST99]

Weierstraß equation $y^2 = x^3 - 20932x - 330140$ with integral points $P = [x, y] = \sum_{i=1}^{7} m_i P_i$, $y > 0$			
$[x, y]$	$(m_1, \ldots, m_7)$	$[x, y]$	$(m_1, \ldots, m_7)$
$[-136, 34]$	$(0, 0, 0, 0, 0, -1, 0)$	$[1336, 48542]$	$(0, 0, 0, 1, 1, 0, 0)$
$[-134, 262]$	$(0, 1, 0, 1, 0, 0, 0)$	$[1546, 60518]$	$(0, 0, 0, -1, 0, -1, 0)$
$[-128, 502]$	$(0, -1, 0, 0, 1, 0, 1)$	$[1862, 80102]$	$(0, 0, 0, -1, -1, 0, -1)$
$[-127, 529]$	$(1, 1, 1, 0, 0, 0, 0)$	$[2849, 151871]$	$(0, 1, 0, 0, 0, 0, 0)$
$[-126, 554]$	$(0, 0, -1, 0, 0, 1, -1)$	$[2966, 161338]$	$(0, 0, 1, 0, 0, 0, 1)$
$[-114, 758]$	$(0, 0, 0, 1, 0, 0, 0)$	$[3161, 177533]$	$(0, -1, -1, 0, -1, -1, 1)$
$[-111, 791]$	$(0, 0, 0, -1, -1, -1, 0)$	$[3288, 188354]$	$(-1, -1, -1, 0, 0, 1, 0)$
$[-95, 895]$	$(-1, 0, 1, 0, 0, 0, 0)$	$[3882, 241702]$	$(-1, 0, 1, 0, 1, 0, 0)$
$[-74, 902]$	$(1, 0, 0, 0, 1, 1, 0)$	$[4753, 327529]$	$(0, 1, 0, 0, -1, -1, -1)$
$[-70, 890]$	$(0, 0, 0, 0, 0, -1, 1)$	$[7778, 685846]$	$(1, 0, -1, 1, 0, 0, 0)$
$[-66, 874]$	$(0, 0, 0, 0, -1, 0, 0)$	$[8154, 736186]$	$(0, 0, -1, 1, -1, 0, 1)$
$[-62, 854]$	$(-1, 0, 0, 0, 0, 0, -1)$	$[8342, 761798]$	$(0, -1, 0, -1, 1, 0, 1)$
$[-32, 554]$	$(0, 0, 0, 1, 0, 0, 1)$	$[10042, 1006202]$	$(1, 1, 1, -1, 0, 0, 0)$
$[-24, 398]$	$(1, 1, 0, 0, 0, 0, 0)$	$[10664, 1101134]$	$(0, 1, 1, 1, 0, -1, 1)$
$[-23, 373]$	$(0, -1, -1, 0, 0, 1, 0)$	$[13030, 1487270]$	$(-1, 0, -1, 1, 0, 0, 0)$
$[-22, 346]$	$(0, 0, 1, 0, 1, 0, 0)$	$[13266, 1527862]$	$(0, -1, 0, -1, 0, 1, 0)$
$[-18, 202]$	$(0, 0, -1, 1, 0, 0, 0)$	$[14217, 1695077]$	$(-1, -1, 0, 1, 0, 0, 1)$
$[-16, 26]$	$(1, 0, 0, 0, 0, 0, 0)$	$[18809, 2579501]$	$(0, 0, 2, -1, 1, 0, 0)$
$[152, 2]$	$(0, 0, 0, 0, 1, 1, -1)$	$[22174, 3301846]$	$(0, -1, 0, -1, 0, -1, 0)$
$[153, 221]$	$(-1, 0, 0, 0, -1, 0, -1)$	$[23714, 3651734]$	$(1, 0, 1, -1, 1, -1, 1)$
$[154, 314]$	$(1, 1, 0, 1, 0, 0, 0)$	$[24030, 3724970]$	$(-1, -1, 0, -1, -1, -1, 1)$
$[158, 554]$	$(0, 0, 1, -1, 0, -1, 0)$	$[24658, 3871946]$	$(-1, -1, 0, -1, 1, 1, -1)$
$[168, 946]$	$(0, 1, 1, 0, 0, 0, 0)$	$[38401, 7525073]$	$(-1, 1, 1, 1, 0, 0, -1)$
$[193, 1679]$	$(0, 0, 0, -1, 0, 1, -1)$	$[46777, 10116877]$	$(1, 1, -1, 2, 0, 1, 0)$
$[194, 1706]$	$(-1, 0, 0, 1, 0, 0, 0)$	$[47656, 10403378]$	$(1, 0, 0, 0, 0, -1, 2)$
$[214, 2234]$	$(0, -1, -1, 0, 0, 0, 1)$	$[49424, 10987654]$	$(1, 0, 0, 0, 2, 1, 0)$
$[230, 2650]$	$(0, 1, 0, 0, 0, 0, -1)$	$[99694, 31477706]$	$(-1, 1, 1, -1, -1, -1, -1)$
$[232, 2702]$	$(0, 0, 1, 0, 0, 0, 0)$	$[138430, 51504490]$	$(1, 1, -1, 0, -1, 0, 0)$
$[280, 3970]$	$(0, 0, -1, 1, -1, 0, 0)$	$[186752, 80704502]$	$(1, 0, 0, 0, -1, 0, 1)$
$[281, 3997]$	$(0, -1, 0, -1, 1, 0, 0)$	$[194617, 85856077]$	$(0, 0, 0, 0, 1, -1, 1)$
$[342, 5702]$	$(1, 0, 0, 0, 0, -1, 1)$	$[200848, 90012154]$	$(1, 0, 0, 0, 1, 2, -1)$
$[346, 5818]$	$(0, 0, 0, 0, 1, 1, 0)$	$[279038, 147399146]$	$(0, -1, 1, 0, 1, -1, 2)$
$[402, 7498]$	$(0, 0, 0, 0, 0, 0, -1)$	$[285918, 152884054]$	$(-2, -1, 0, -1, -1, 0, -1)$
$[406, 7622]$	$(-1, 0, 0, 0, -1, 0, 0)$	$[517006, 371743478]$	$(0, -1, -2, 0, -1, 1, 0)$
$[528, 11654]$	$(0, 1, 0, 1, 0, 1, -1)$	$[594488, 458368354]$	$(1, 2, 1, 0, 0, -1, 0)$
$[529, 11689]$	$(0, 0, 1, -1, 0, -1, 1)$	$[1063216, 1096307162]$	$(-1, -2, -1, 0, 1, 0, 1)$
$[656, 16378]$	$(-1, 0, 1, -1, 0, 0, -1)$	$[1173142, 1270649798]$	$(1, 1, 2, 0, 0, -1, 1)$
$[672, 17002]$	$(1, 0, -1, 0, 0, 1, 0)$	$[7160514, 19160917334]$	$(0, 1, -1, 0, 0, 0, -1)$

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Weierstraß equation $y^2 = x^3 - 20932x - 330140$ with integral points $P = [x, y] = \sum_{i=1}^{7} m_i P_i$, $y > 0$			
$[x, y]$	$(m_1, \ldots, m_7)$	$[x, y]$	$(m_1, \ldots, m_7)$
$[722, 18998]$	$(1, 1, 1, 0, 1, 0, 0)$	$[8293712, 23884899898]$	$(1, 0, 0, 0, -1, -2, 1)$
$[774, 21146]$	$(0, -1, 0, 0, 1, 1, 0)$	$[11457838, 38784126646]$	$(1, 0, 0, -2, 0, 1, -1)$
$[878, 25654]$	$(-1, -1, 0, 0, 0, -1, 1)$	$[14287592, 54005570702]$	$(1, 0, 0, 0, 1, 0, -1)$
$[930, 28010]$	$(0, 1, 1, 0, -1, -1, 0)$	$[63472528, 505683395846]$	$(0, 1, -1, 2, -2, 0, 0)$
$[984, 30526]$	$(-1, 0, -1, 0, -1, 0, -1)$	$[146288721, 1769360242057]$	$(0, 0, 0, 0, -2, -2, 2)$
$[1137, 38023]$	$(1, 0, 0, 1, 0, 0, 1)$	$[544894592, 12719462124298]$	$(0, 1, -1, 2, 0, 2, -2)$

3 Example 5: Siksek

The points P_i, for $i = 1, \ldots, 8$, in the table below are the elements of the c_1-optimal basis $\mathcal{B}_2$ in TABLE 4 of [ST99]. Further, $Q = [1402932, -701466]$ generates the torsion group, and $\varepsilon \in \{0, 1\}$.

Table 4: All solutions to Siksek's equation (19) of [ST99]

Weierstraß equation

$$y^2 + xy = x^3 - 5818216808130x + 5401285759982786436$$

with integral points $P = [x, y] = \sum_{i=1}^{8} m_i P_i + \varepsilon Q$;

of the additive inverses $[x, y]$, $[x, -y - x]$ only one is listed

ε	$[x, y]$	$(m_1, \ldots, m_8)$
0	$[-2520768, 2013726114]$	$(1, 0, 0, 0, 0, 0, 0, 0)$
0	$[-2433318, 2270818884]$	$(0, 0, 1, 0, 0, 0, 1, 1)$
0	$[-393324, 2762239638]$	$(0, 0, 0, 0, 0, 1, 1, 1)$
0	$[242106, 2001591138]$	$(0, 0, 1, 0, 1, 0, 0, 1)$
0	$[975216, 808674546]$	$(0, 0, 0, 0, -1, 1, 1, 0)$
0	$[1145136, 489626526]$	$(0, 0, 1, 0, 0, 0, 0, 0)$
0	$[1284264, 216935646]$	$(0, -1, -1, 0, 0, 0, 0, 0)$
0	$[1365048, 51389034]$	$(-1, 1, 0, 0, 0, 0, 0, 0)$
0	$[1368480, 43776066]$	$(0, 0, 0, 0, 0, 0, -1, 0)$
0	$[1404150, 9858594]$	$(0, 0, 0, 0, 0, 1, 0, 1)$
0	$[1410240, 28567074]$	$(0, -1, 0, 0, 0, 0, 0, 0)$
0	$[1421184, 53932386]$	$(0, 1, 0, -1, 0, 0, 0, 0)$
0	$[1437384, 88804830]$	$(0, 0, 0, 0, -1, 1, 0, 0)$
0	$[1704648, 659967834]$	$(0, 0, 0, 0, 0, 1, -1, 0)$
0	$[4740024, 9180268266]$	$(0, 0, 0, 0, -1, 0, 1, 0)$
0	$[6227598, 14512139184]$	$(0, 0, 0, 0, 1, 0, 0, 1)$
0	$[6625866, 16050970146]$	$(0, 1, 0, 0, 0, -1, 0, -1)$
0	$[7028688, 17652683154]$	$(0, -1, -1, 0, 0, 0, 1, 0)$
0	$[8910264, 25704887646]$	$(0, 0, 1, 0, 0, -1, 0, 0)$
0	$[16306128, 65154428574]$	$(0, -1, 0, 0, 1, -1, 0, 0)$
0	$[22282920, 104582643834]$	$(1, 0, -1, -1, 0, 0, 0, 0)$
1	$[-2770950, 498846306]$	$(0, 0, 1, 0, 0, -1, 1, 0)$
1	$[-2397676, 2360776310]$	$(1, 0, -1, -1, 0, 0, 1, 0)$
1	$[-1106022, 3238358148]$	$(0, 0, 0, 0, 0, 0, 1, 0)$
1	$[-878730, 3136583784]$	$(1, -1, 0, 0, 0, 0, 0, 0)$
1	$[674526, 1335195138]$	$(0, 1, 1, 0, 0, 0, 0, 0)$
1	$[1067634, 637035330]$	$(0, 0, -1, 0, 0, 0, 0, 0)$
1	$[1402932, -701466]$	$(0, 0, 0, 0, 0, 0, 0, 0)$
1	$[1998516, 1324026246]$	$(1, 0, 1, 0, 0, 0, 0, 0)$
1	$[3911886, 6517561026]$	$(0, 0, 0, 0, 1, -1, 0, 0)$
1	$[13230858, 47370142164]$	$(0, 1, 0, 0, 0, 0, 0, 0)$

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Weierstraß equation $y^2 + xy = x^3 - 5818216808130x + 5401285759982786436$ with integral points $P = [x, y] = \sum_{i=1}^{6} m_i P_i + \varepsilon Q$; of the additive inverses $[x, y]$, $[x, -y - x]$ only one is listed		
ε	$[x, y]$	$(m_1, \ldots, m_8)$
1	$[72370488, 615288009054]$	$(0, 0, 0, 0, 0, -1, 0, -1)$
1	$[246339954, 3866052130626]$	$(-1, 1, 1, 1, 0, 0, 0, 0)$
1	$[1266482286, 45070439266626]$	$(0, 0, 1, 1, 0, 0, -1, 0)$
1	$[1486191694597896, 57294408386916566732766]$	$(0, 0, -1, 0, -1, -1, 0, -1)$

References

[ST99] Roel J. Stroeker and Nikos Tzanakis, "On the Elliptic Logarithm Method for Elliptic Diophantine Equations: Reflections and an Improvement", *Experimental Mathematics* **8**:2 (1999), 135–149.